NURBS Curves

A *p*-th degree ***N***on-***U***niform ***R***ational ***B***-***S***pline (NURBS) curve is defined by:

$$\boldsymbol{C}(u) = \frac{\sum_{i=0}^{n} N_{i,p}(u)\, w_i \boldsymbol{P}_i}{\sum_{i=0}^{n} N_{i,p}(u)\, w_i} \quad , \quad 0 \le u \le 1$$

where the $\{\mathbf{P}_i\}$ are the *control points*, the $\{w_i\}$ are the *weights*, and the $\{N_{i,p}(u)\}$ are the p-th degree B-spline basis functions defined on the nonperiodic (and nonuniform) knot vector,

$$U = \{\underbrace{0, \ldots, 0}_{p+1}, u_{p+1}, \ldots, u_{m-p-1}, \underbrace{1, \ldots, 1}_{p+1}\}$$

Setting,

$$R_{i,p}(u) = \frac{N_{i,p}(u)\, w_i}{\sum_{i=0}^{n} N_{i,p}(u)\, w_i}$$

allows us to write the NURBS curve as,

$$\boldsymbol{C}(u) = \sum_{i=0}^{n} R_{i,p}(u)\, \boldsymbol{P}_i \qquad , \quad 0 \le u \le 1$$

The properties of $R_{i,p}$ and $\boldsymbol{C}(u)$ follow in a similar fashion from those for $N_{i,p}$ and the non-rational form of $\boldsymbol{C}(u)$

As was the case for rational Bezier curves, homogeneous coordinates are used to represent NURBS curves and surfaces. Let H be the perspective map as before. For a given set of control points $\{\boldsymbol{P}_i\}$, and weights $\{w_i\}$, construct the weighted control points: $\boldsymbol{P}^w{}_i = (w_i x_i, w_i y_i, w_i z_i, w_i)$. Then define the nonrational (piecewise polynomial) B-spline curve in 4D:

$$\boldsymbol{C}^w(u) = \sum_{i=0}^{n} N_{i,p}(u)\,\boldsymbol{P}^w{}_i \qquad , \quad 0 \le u \le 1$$

Applying the perspective map, *H*, to $\boldsymbol{C}^w(u)$ yields the corresponding rational B-spline curve (piecewise rational in 3D space):

$$
\begin{aligned}
\boldsymbol{C}(u) &= H\{\boldsymbol{C}^w(u)\} \\
&= H\left\{\sum_{i=0}^{n} N_{i,p}(u)\,\boldsymbol{P}^w{}_i\right\} \\
&= \sum_{i=0}^{n} R_{i,p}(u)\,\boldsymbol{P}_i
\end{aligned}
$$

Algorithm A4.1 computes a point on a rational B-spline curve at a fixed *u*-value. It assumes weighted control point array **`Pw`** (as do all future algorithms), i.e., **`Pw[i]`** = ($w_i x_i$, $w_i y_i$, $w_i z_i$, w_i). **`Cw`** denotes the 4D point on $\boldsymbol{C}^w(u)$, and C, the 3D point on $\boldsymbol{C}(u)$ (output).

See algorithm A4.1

Derivatives of NURBS Curves

It would be helpful to use the algorithms already developed for derivatives of nonrational curves. Thus we develop formulas that express the derivatives of $\boldsymbol{C}(u)$ in terms of the derivatives of $\boldsymbol{C}^{w}(u)$. Let,

$$\boldsymbol{C}(u) = \frac{w(u)\,\boldsymbol{C}(u)}{w(u)} = \frac{\boldsymbol{A}(u)}{w(u)}$$

where $\boldsymbol{A}(u)$ is a vector valued function whose coordinates are the first three coordinates of $\boldsymbol{C}^{w}(u)$. Then,

$$\boldsymbol{C}'(u) = \frac{w(u)\boldsymbol{A}'(u) - w'(u)\boldsymbol{A}(u)}{(w(u))^2}$$

$$= \frac{w(u)\boldsymbol{A}'(u) - w'(u)w(u)\boldsymbol{C}(u)}{(w(u))^2}$$

$$= \frac{\boldsymbol{A}'(u) - w'(u)\boldsymbol{C}(u)}{w(u)}$$

Since $\boldsymbol{A}(u)$ and $w(u)$ represent the coordinates of $\boldsymbol{C}^{w}(u)$, we obtain their first derivatives by using the nonrational formulas. For higher order derivatives, we differentiate $\boldsymbol{A}(u)$ using Leibnitz' rule:

$$\boldsymbol{A}^{(k)}(u) = (w(u)\boldsymbol{C}(u))^{(k)} = \sum_{i=0}^{k} \binom{k}{i} w^{(i)}(u)\boldsymbol{C}^{(k-i)}(u)$$

$$\boldsymbol{A}^{(k)}(u) = w(u)\,\boldsymbol{C}^{(k)}(u) + \sum_{i=1}^{k} \binom{k}{i} w^{(i)}(u)\,\boldsymbol{C}^{(k-i)}(u)$$

from which we obtain:

$$\boldsymbol{C}^{(k)}(u) = \frac{\boldsymbol{A}^{(k)}(u) - \sum_{i=1}^{k} \binom{k}{i} w^{(i)}(u)\, \boldsymbol{C}^{(k-i)}(u)}{w(u)}$$

This formula gives the *k*-th derivative of ***C***(*u*) in terms of the *k*-th derivative of ***A***(*u*) and the first through (*k* - 1)-th derivative of *w*(*u*).

The derivatives $\boldsymbol{A}^{(k)}(u)$ and $w^{(i)}(u)$ are obtained using either algorithm A3.2 or algorithm A3.4.

Exercise:

Consider the quadratic rational Bezier circular arc given by:

$U = \{0, 0, 0, 1, 1, 1\}$, $\boldsymbol{P}_i = \{ (1,0), (1,1), (0,1) \}$, and $w_i = \{ 1, 1, 2 \}$. Compute the first and second derivatives at $u = 0$, and $u = 1$.

Now assume that u is fixed, and that the 0-th through the d-th derivatives of $\boldsymbol{A}(u)$ and $w(u)$ have been computed and loaded into the arrays **`Aders`** and **`wders`**, respectively; i.e., $\boldsymbol{C}^w(u)$ has been differentiated and its coordinates separated off into **`Aders`** and **`wders`**.

An algorithm to compute the point $\boldsymbol{C}(u)$ and the derivatives $\boldsymbol{C}^{(k)}(u)$, $1 \le k \le d$, follows. The curve point is returned in **`CK[0]`**, and the k-th derivative in **`CK[k]`**.

See algorithm A4.2

NURBS Surfaces

A NURBS surface of degree *p* in the *u*-direction and degree *q* in the *v*-direction is a bivariate vector-valued piecewise rational polynomial of the form:

$$\boldsymbol{S}(u, v) = \frac{\sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,p}(u) N_{j,q}(v) w_{ij} \boldsymbol{P}_{ij}}{\sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,p}(u) N_{j,q}(v) w_{ij}}$$

for $0 \leq u, v \leq 1$

the $\{\boldsymbol{P}_{ij}\}$ form a bidirectional control net, the $\{w_{ij}\}$ are the weights, and the $\{N_{i,p}(u)\}$ and $\{N_{j,q}(u)\}$ are the B-spline basis functions defined on the nonperiodic (and nonuniform) knot vectors,

$$U = \{\underbrace{0, \ldots, 0}_{p+1}, u_{p+1}, \cdots, u_{r-p-1}, \underbrace{1, \ldots, 1}_{p+1}\}$$

and,

$$V = \{\underbrace{0, \ldots, 0}_{q+1}, v_{q+1}, \ldots, v_{s-q-1}, \underbrace{1, \ldots, 1}_{q+1}\}$$

where,

$$r = n + p + 1 \qquad \text{and} \qquad s = m + q + 1$$

Introducing the piecewise rational basis functions:

$$R_{i,j}(u,v) = \frac{N_{i,p}(u)N_{j,q}(v)w_{ij}}{\sum_{k=0}^{n}\sum_{l=0}^{m}N_{k,p}(u)N_{l,q}(v)w_{kl}}$$

the surface can be written in the form:

$$\boldsymbol{S}(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} R_{i,j}(u, v)\boldsymbol{P}_{ij}$$

for $0 \le u \le 1$

The properties of $\boldsymbol{S}(u,v)$ follow in a similar fashion from those for the non-rational form of $\boldsymbol{S}(u,v)$

Algorithm A3.5 can be adapted to compute a point on a rational B-spline surface by simply allowing the array **P** to contain weighted control points (use **Pw**), accumulating the 4D surface point in **Sw**, and inserting a line to accomplish the perspective projection.

See algorithm A4.3

Derivatives of a NURBS Surface

In a manner similar to that for curves, the formulation of derivatives of a NURBS surface ***S***(*u, v*) is derived in terms of those for $\boldsymbol{S}^w(u, v)$. Thus, let

$$\boldsymbol{S}(u, v) = \frac{w(u, v)\,\boldsymbol{S}(u, v)}{w(u, v)} = \frac{\boldsymbol{A}(u, v)}{w(u, v)}$$

where $\boldsymbol{A}(u, v)$ is the numerator of $\boldsymbol{S}(u, v)$. Then,

$$\boldsymbol{S}_\alpha(u, v) = \frac{\boldsymbol{A}_\alpha(u, v) - w_\alpha(u, v)\,\boldsymbol{S}(u, v)}{w(u, v)}$$

where α denotes either u or v.

In general,

$$\boldsymbol{A}^{(k,l)} = \left((w\boldsymbol{S})^{k} \right)^{l}$$

$$= \left(\sum_{i=0}^{k} \binom{k}{i} w^{(i,0)} \boldsymbol{S}^{(k-i,0)} \right)^{l}$$

$$= \sum_{i=0}^{k} \binom{k}{i} \sum_{j=0}^{l} \binom{l}{j} w^{(i,j)} \boldsymbol{S}^{(k-i,l-j)}$$

$$
\begin{aligned}
\boldsymbol{A}^{(k,l)} = {} & w^{(0,0)}\boldsymbol{S}^{(k,l)} \\
& + \sum_{i=1}^{k}\binom{k}{i} w^{(i,0)}\boldsymbol{S}^{(k-i,l)} \\
& + \sum_{j=1}^{l}\binom{l}{j} w^{(0,j)}\boldsymbol{S}^{(k,l-j)} \\
& + \sum_{i=1}^{k}\binom{k}{i}\sum_{j=1}^{l}\binom{l}{j} w^{(i,j)}\boldsymbol{S}^{(k-i,l-j)}
\end{aligned}
$$

and it follows that:

$$\boldsymbol{S}^{(k,l)} = \frac{1}{w}\left(\boldsymbol{A}^{(k,l)} - \sum_{i=1}^{k}\binom{k}{i} w^{(i,0)}\boldsymbol{S}^{(k-i,l)} - \sum_{j=1}^{l}\binom{l}{j} w^{(0,j)}\boldsymbol{S}^{(k,l-j)} - \sum_{i=1}^{k}\binom{k}{i}\sum_{j=1}^{l}\binom{l}{j} w^{(i,j)}\boldsymbol{S}^{(k-i,l-j)}\right)$$

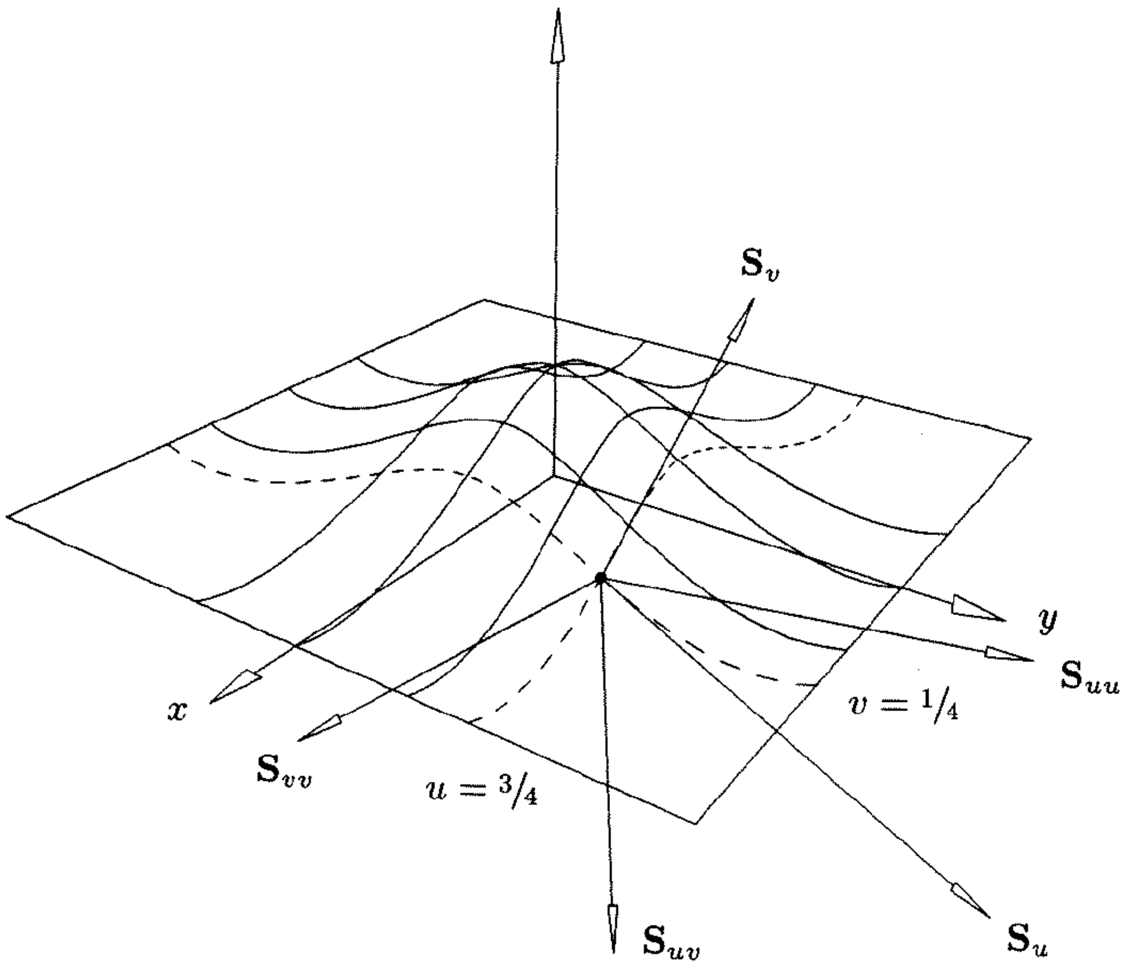
from which we get the following formulas:

$$\boldsymbol{S}_{uv} = \frac{\boldsymbol{A}_{uv} - w_{uv}\boldsymbol{S} - w_u\boldsymbol{S}_v - w_v\boldsymbol{S}_u}{w}$$

$$\boldsymbol{S}_{uu} = \frac{\boldsymbol{A}_{uu} - 2w_u\boldsymbol{S}_u - w_{uu}\boldsymbol{S}}{w}$$

$$\boldsymbol{S}_{vv} = \frac{\boldsymbol{A}_{vv} - 2w_v\boldsymbol{S}_v - w_{vv}\boldsymbol{S}}{w}$$

Example:



$\mathbf{S}_v$
y
$\mathbf{S}_{uu}$
x
$v = 1/4$
$\mathbf{S}_{vv}$
$u = 3/4$
$\mathbf{S}_{uv}$
$\mathbf{S}_u$