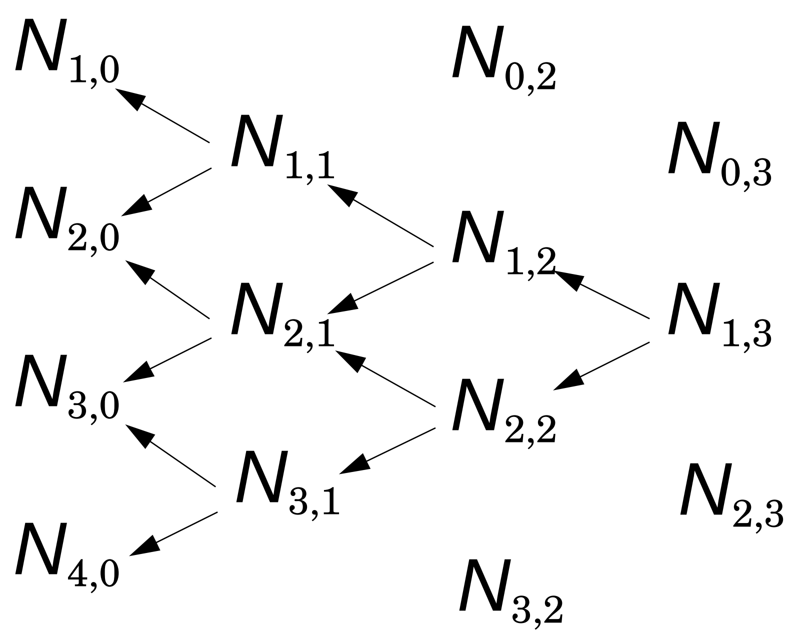
Algorithms for computing a single basis function $N_{i,p}(u)$ and its derivatives take advantage of Property P2.1, i.e., a triangular table,



B-spline Curves and Surfaces

Consider a *p*-th degree nonrational B-spline curve defined by:

$$\boldsymbol{C}(u) = \sum_{i=0}^{n} N_{i,p}(u)\,\boldsymbol{P}_i \quad , \quad a \le u \le b$$

where the { $\boldsymbol{P}_i$ } are the *control points*, and the { $N_{i,p}(u)$ } are *p*-th degree B-spline basis functions defined on the nonperiodic (and generally nonuniform) knot vector:

$$U = \{\underbrace{0, \ldots, 0}_{p+1}, u_{p+1}, \ldots u_{m-p-1}, \underbrace{1, \ldots, 1}_{p+1}\}$$

(with total m + 1 knots).

As with Bezier curves, the polygon formed by the { $\boldsymbol{P}_i$ } is called the *control polygon*.

Three steps are required to compute a point on a B-spline curve at a fixed *u*-value:

1. Find the knot span in which u lies (Algorithm A2.1)

2. Compute the nonzero basis functions (Algorithm A2.2)

3. Multiply the values of the nonzero basis functions with the corresponding control points.

This suggests a simple algorithm

See algorithm A3.1

Properties of B-spline Curves

P3.1: If $n = p$ and $U = \{0, ..., 0, 1, ..., 1\}$, then $\mathbf{C}(u)$ is a Bezier curve.

P3.2: $\mathbf{C}(u)$ is a piecewise polynomial curve (since the $N_{i,p}(u)$ are piecewise polynomials). The degree, p, number of control points, $n + 1$, and the number of knots, $m + 1$ are related by the formula: $m = n + p + 1$

P3.3: Endpoint interpolation: $\boldsymbol{C}(0) = \boldsymbol{P}_0$ and $\boldsymbol{C}(1) = \boldsymbol{P}_n$.

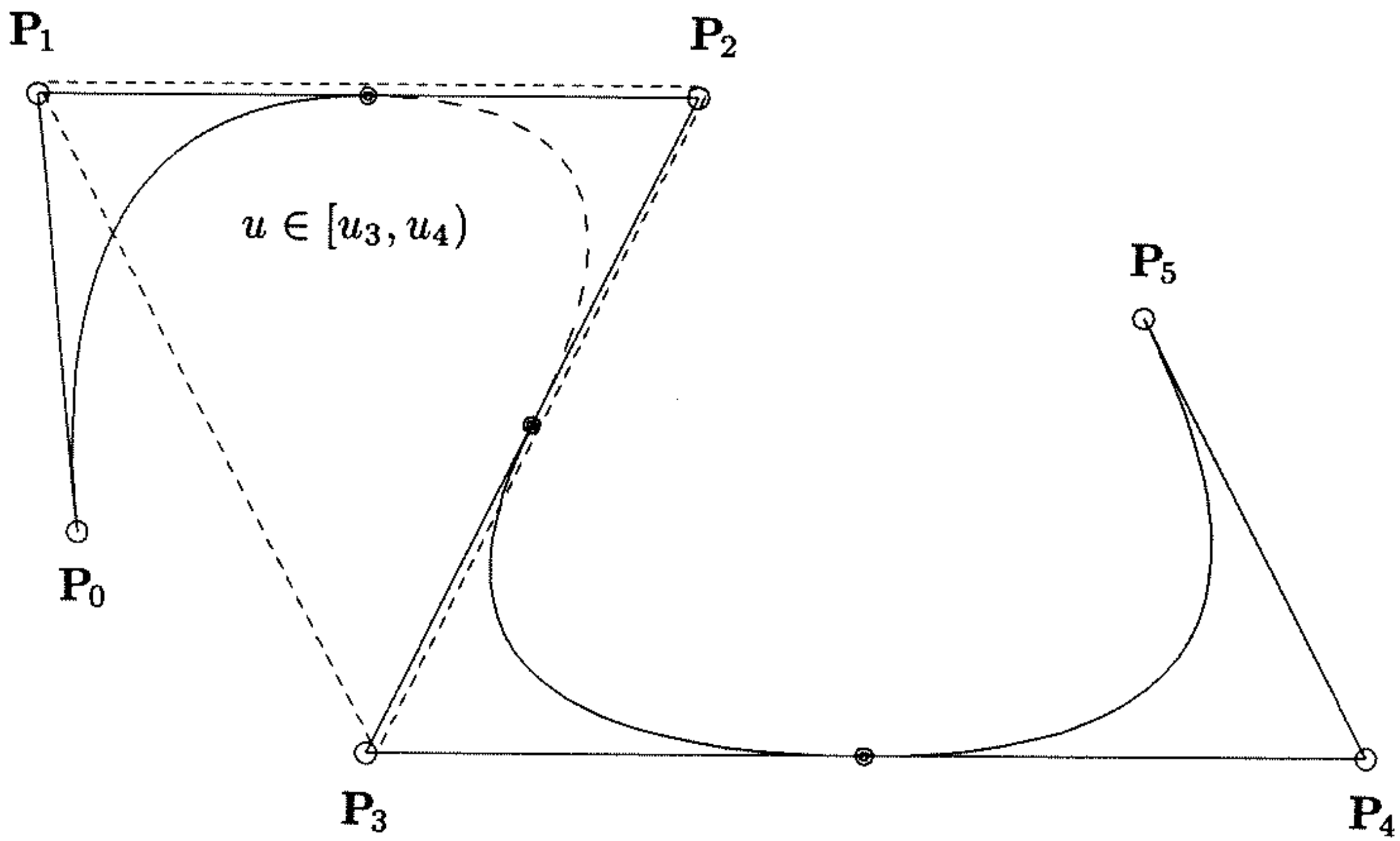
P3.4: Affine invariance: an affine transformation is applied to the curve by applying it to the control points. Let **r** be a point in E^3 (3D Euclidean space). An affine transformation denoted by Φ, maps E^3 into E^3 and has the form:

$$\Phi(\mathbf{r}) = A\mathbf{r} + \mathbf{v},$$

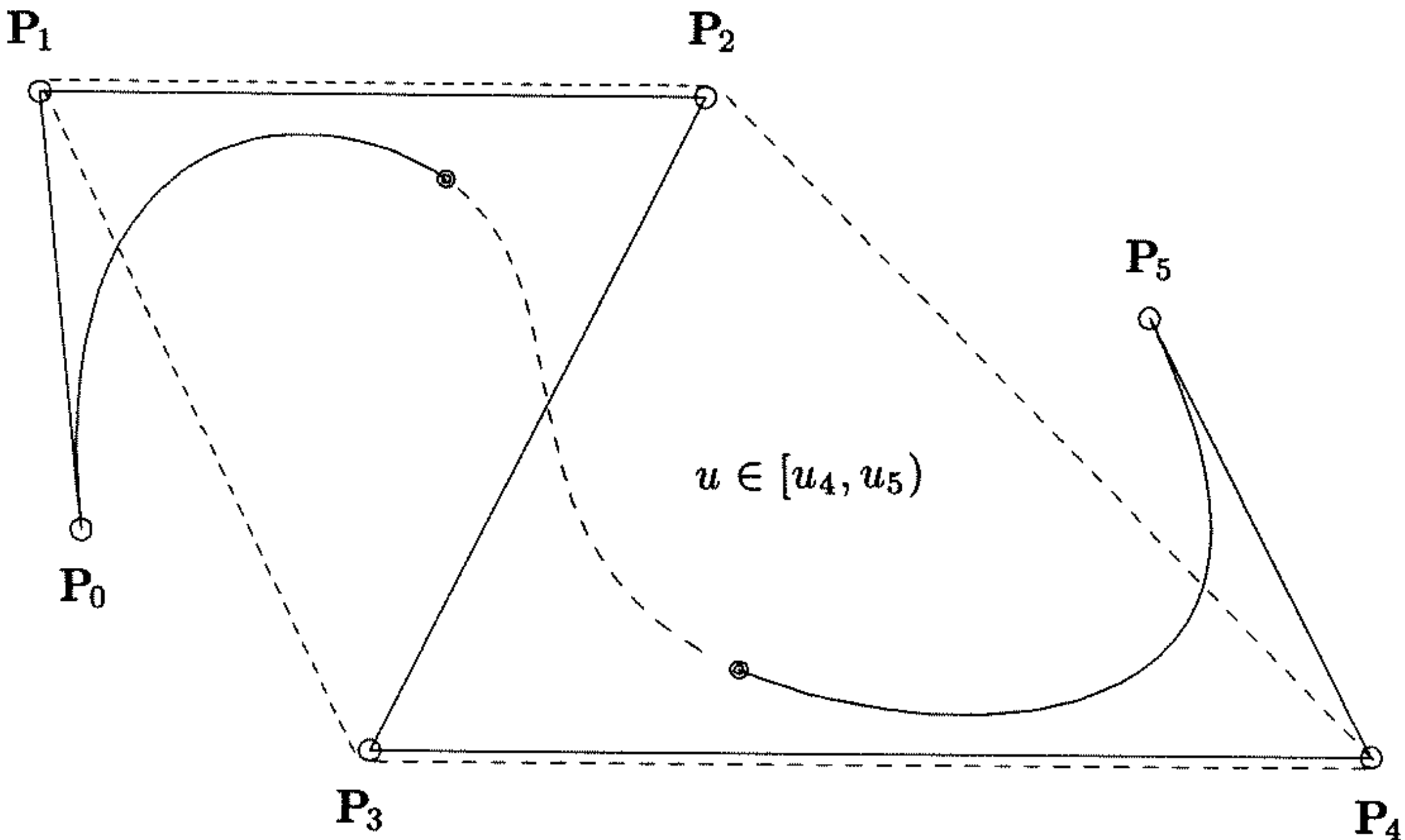
where A is a 3×3 matrix and **v** is a vector.

Note: Affine transformations include translations, rotations, scaling and shears. The affine invariance property for B-spline curves follows from the partition of unity property of $N_{i,p}(u)$

P3.5: Strong convex hull property: the curve is contained in the convex hull of its control polygon. In fact if $u \in [u_i, u_{i+1})$, $p \leq i < m - p - 1$, then $\boldsymbol{C}(u)$ is in the convex hull of the control points, $\boldsymbol{P}_{i-p}, ..., \boldsymbol{P}_i$. This follows from the non-negativity and partition of unity properties of $N_{i,p}(u)$ (and property P2.2)

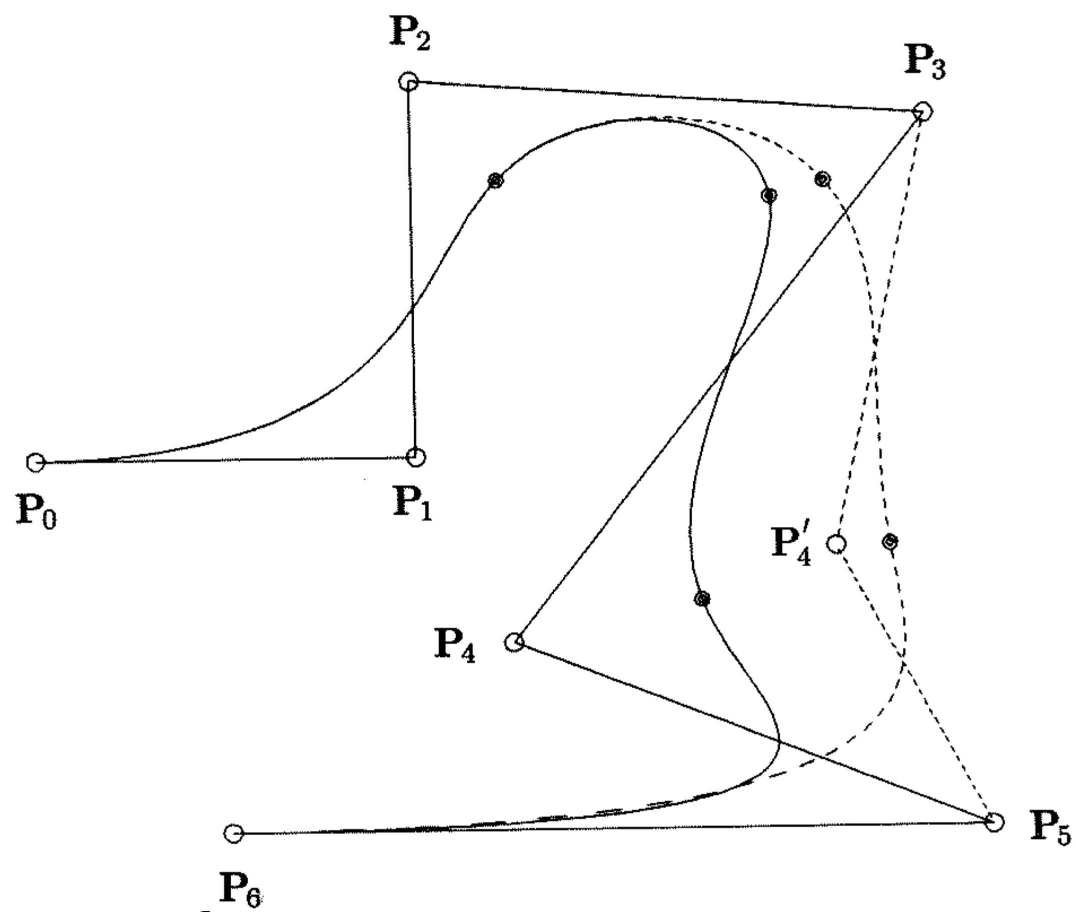


$p = 2$, $U = \{ 0, 0, 0, .25, .5, .75, 1\ 1\ 1\}$



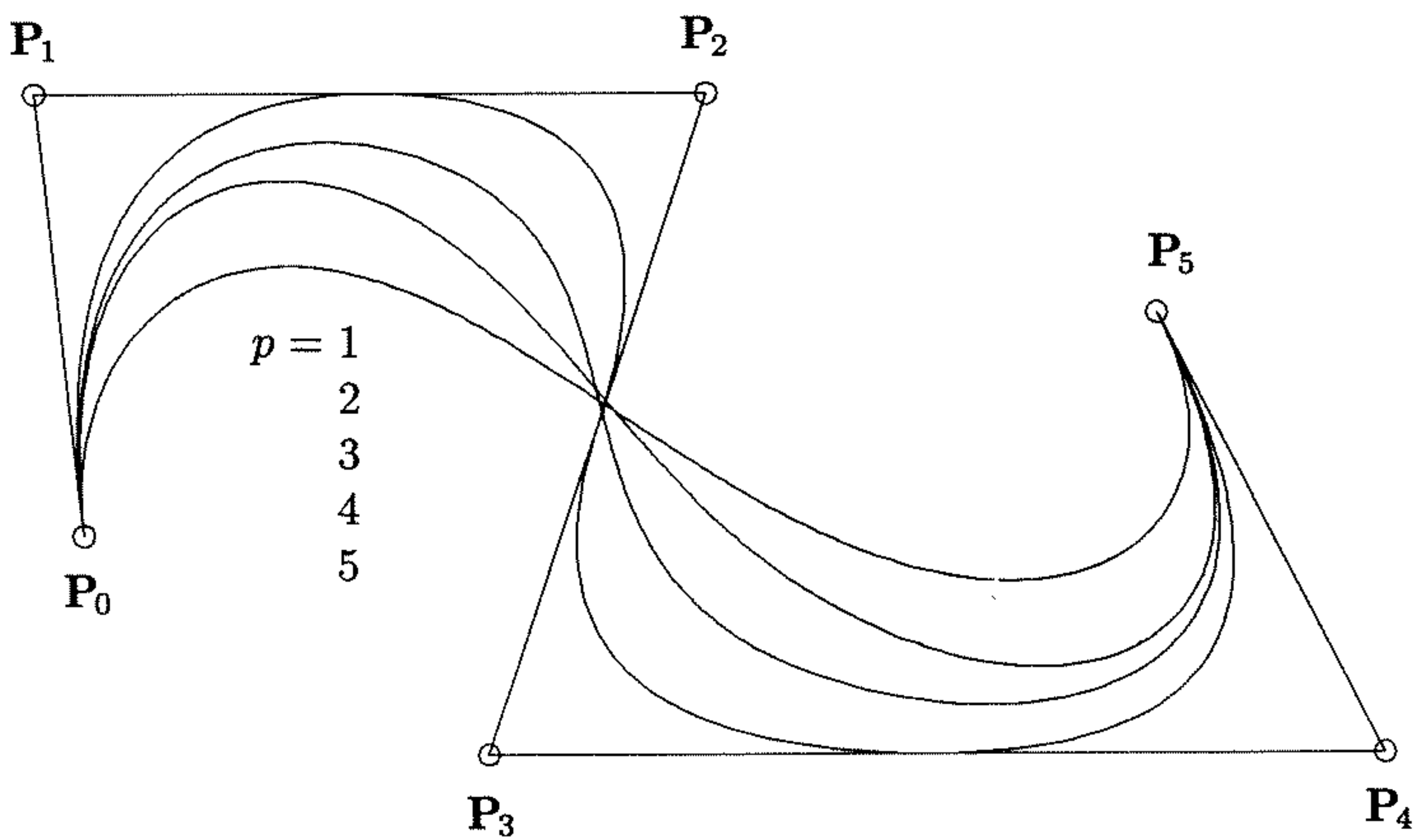
p = 3, U = { 0, 0, 0, 0, .33, .67, 1, 1, 1, 1}

P3.6: Local modification scheme: moving $\boldsymbol{P}_i$ changes $\boldsymbol{C}(u)$ only in the interval $[u_i, u_{i+p+1})$. This follows from the fact that $N_{i,p}(u) \notin [u_i, u_{i+1})$



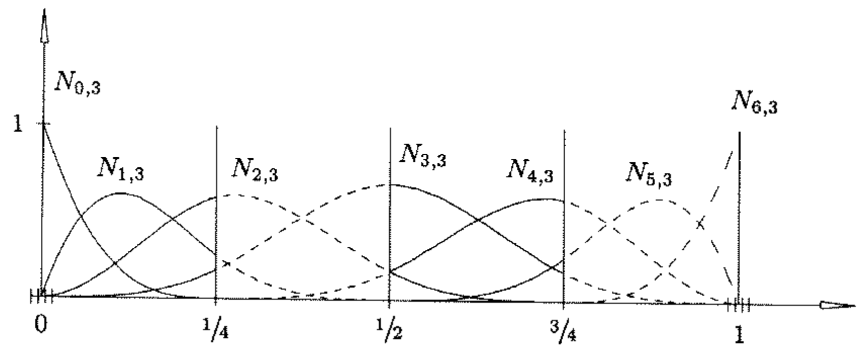
$p = 3, U = \{0, 0, 0, 0, .25, .5, .75, 1, 1, 1, 1\}$

P3.7: The control point polygon represents a piecewise linear approximation to the curve. As a general rule, the lower the degree, the closer the B-spline curve follows its control polygon



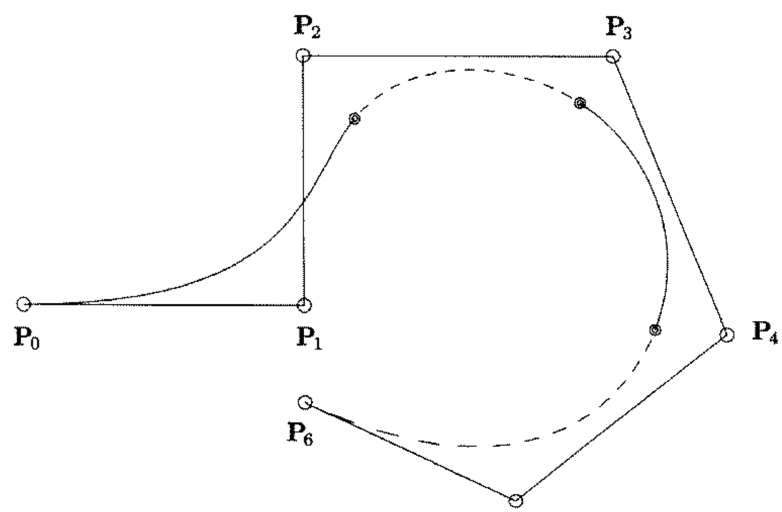
P1
P2
P5
p = 1
2
3
4
5
P0
P3
P4

P3.8: Moving along the curve from $u = 0$ to $u = 1$, the $N_{i,p}(u)$ functions act like switches. As u moves past a knot, one of the $N_{i,p}(u)$ (and hence, the corresponding $\boldsymbol{P}_i$) switches off, and the next one switches on.



$N_{0,3}$
1
$N_{1,3}$
$N_{2,3}$
$N_{3,3}$
$N_{4,3}$
$N_{5,3}$
$N_{6,3}$
0
1/4
1/2
3/4
1

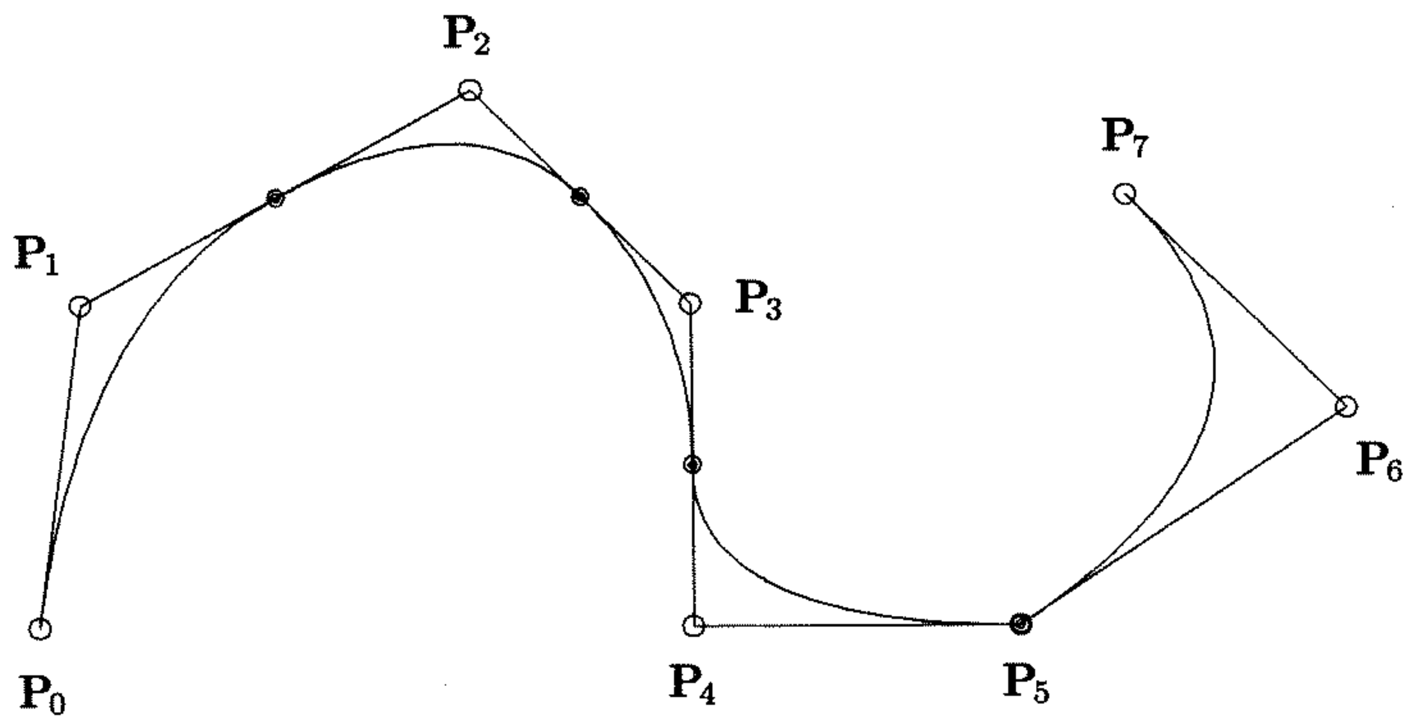
(a)



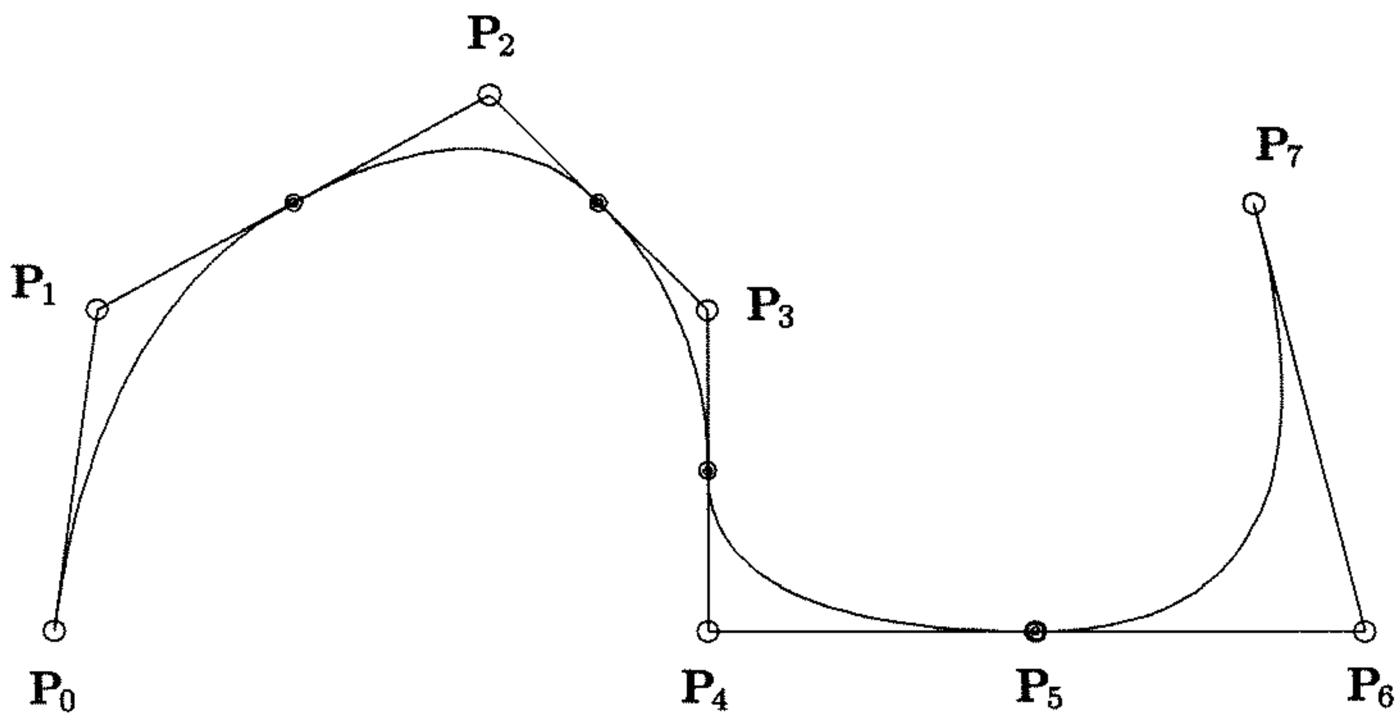
$\mathbf{P}_2$
$\mathbf{P}_3$
$\mathbf{P}_0$
$\mathbf{P}_1$
$\mathbf{P}_4$
$\mathbf{P}_6$

P3.9: Variation diminishing property: no plane has more intersections with the curve than the control polygon.

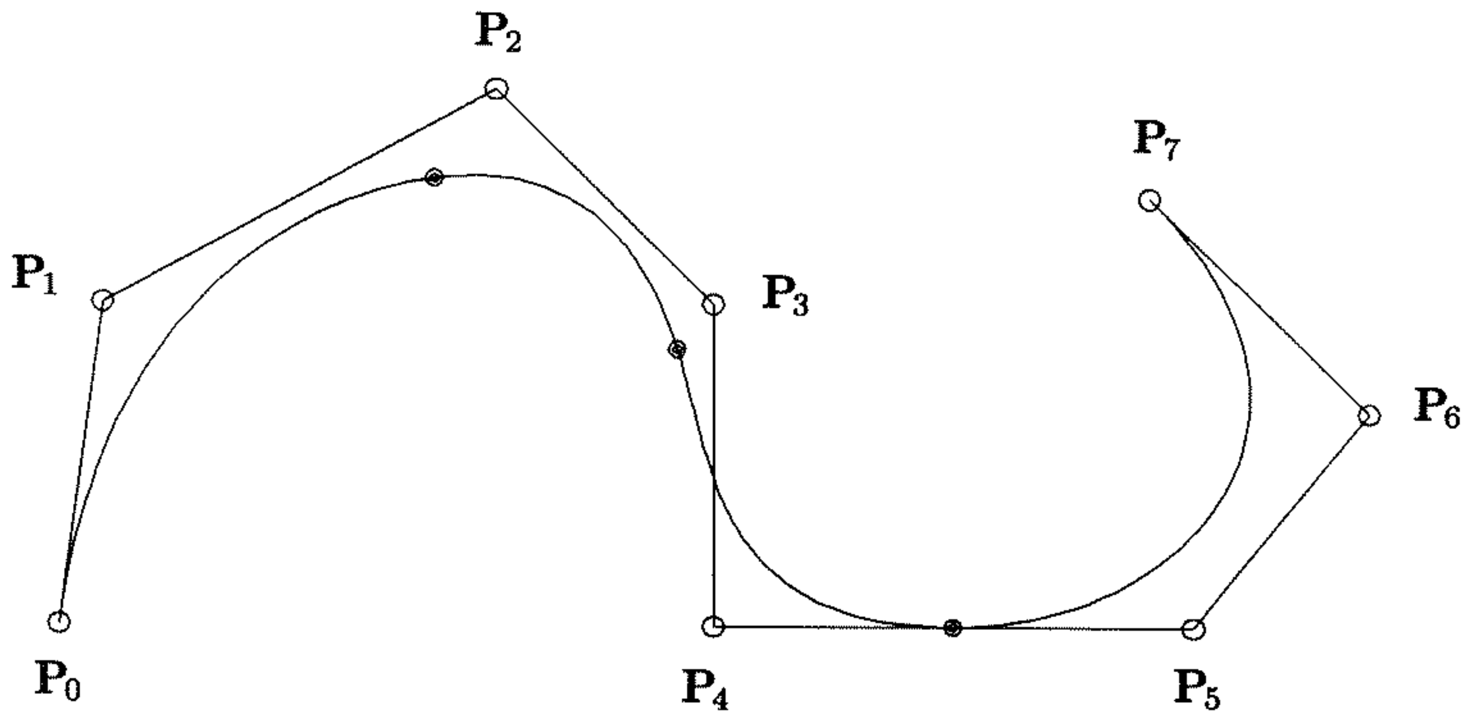
P3.10: $\boldsymbol{C}(u)$ is infinitely differentiable in the interior of knot intervals, and it is at least $p - k$ times continuously differentiable at a knot of multiplicity k.



$$p = 2,\ U = \{0, 0, 0, .2, .4, .6, .8, .8, 1, 1, 1\}$$

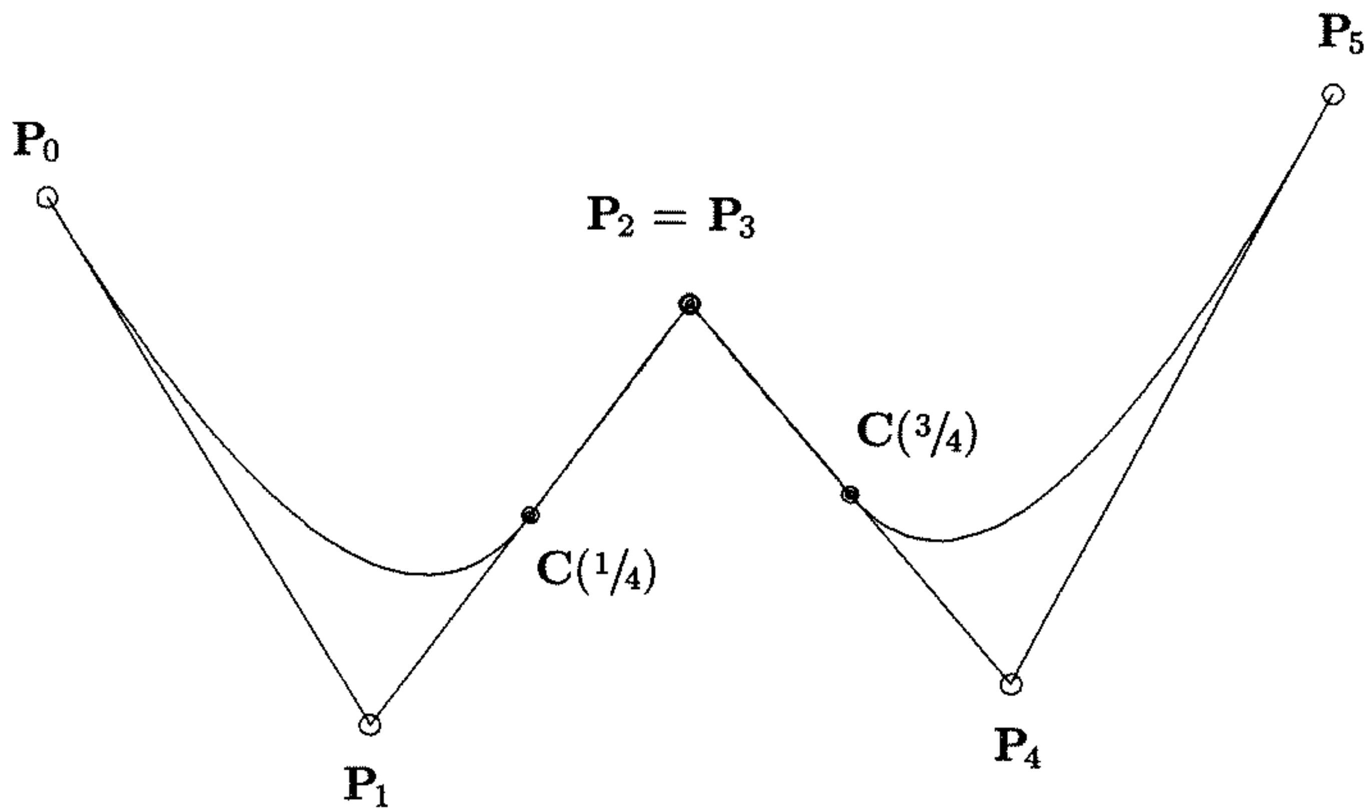


$p = 2$, $U = \{0, 0, 0, .2, .4, .6, .8, .8, 1, 1, 1\}$



$p = 3,\ U = \{0,0,0,0,\ .25,\ .5,\ .75,\ .75,\ 1,1,1,1\}$

P3.11: It is possible (and sometimes useful) to use multiple (coincident) control points. Compare the effect to multiple knots and note that the distinction is due to the convex hull property.



$p = 2$, $U = \{0, 0, 0, .25, .5, .75, 1, 1, 1\}$