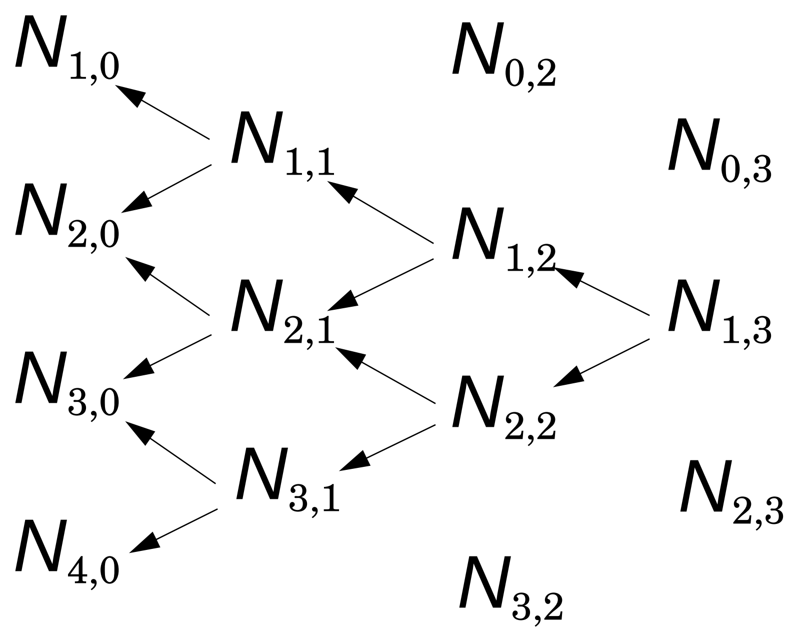
Important Properties of B-spline Basis Functions

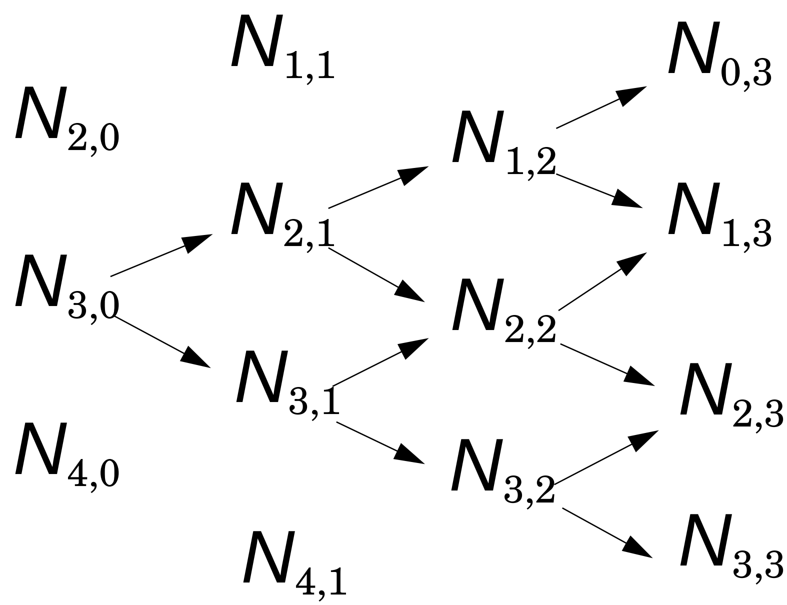
P2.1 $N_{i,p}(u) = 0$ if u is outside the interval $[u_i, u_{i+p+1})$ (local support property).

For example, note that $N_{1,3}$ is a combination of $N_{1,0}$, $N_{2,0}$, $N_{3,0}$, and $N_{4,0}$ Thus, $N_{1,3}$ is non-zero only on the interval $u \in [u_1, u_5]$.



P2.2 In any given knot span, $[u_j, u_{j+1})$, at most $p + 1$ of the $N_{i,p}$ are non-zero, namely, the functions: $N_{j-p,p}, \ldots, N_{j,p}$.

For example, on $[u_3, u_4)$ the only non-zero, 0-th degree function is $N_{3,0}$. Hence the only cubic functions not zero on $[u_3, u_4)$ are $N_{0,3}, \ldots, N_{3,3}$.



P2.3 $N_{i,p}(u) \geq 0$ for all i, p, and u (Non-negativity). Can be proven by induction using P2.1.

P2.4 For arbitrary knot span, $[u_i, u_{i+1})$,

$$\sum_{j=i-p}^{i} N_{j,p}(u) = 1 \text{ for all } u \in [u_i, u_{i+1})$$

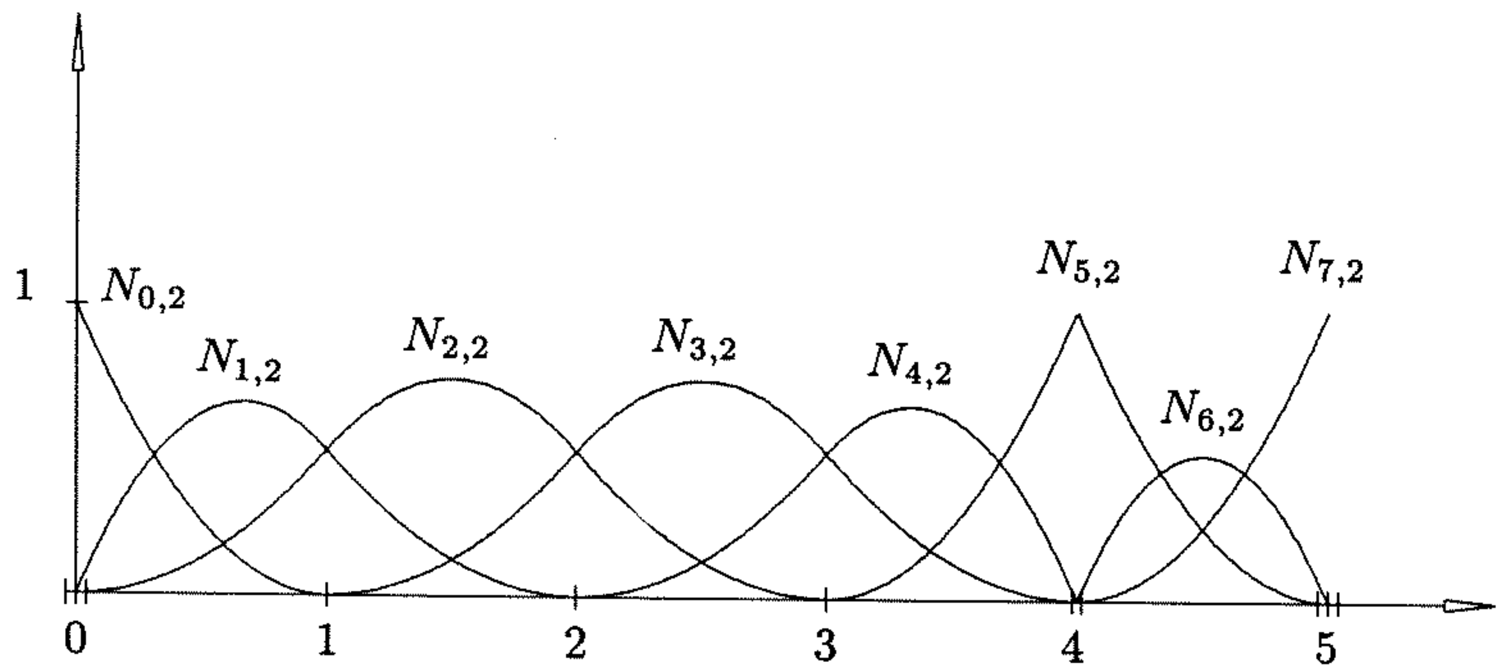
(Partition of unity)

P2.5 All derivatives of $N_{i,p}(u)$ exist in the interior of a knot span (where it is a polynomial). At a knot, $N_{i,p}(u)$ is p - k times continuously differentiable, where k is the multiplicity of the knot. Hence, increasing the degree increases continuity, and increasing knot multiplicity decreases continuity.

P 2.6 Except for the case of $p = 0$, $N_{i,p}(u)$ attains exactly one maximum value.

The concept of multiple knots is *important*.

Consider the example, $p = 2$, and $U = \{ 0, 0, 0, 1, 2, 3, 4, 4, 5, 5, 5 \}$. The basis function, $N_{7,2}$ are plotted below, with knot multiplicity indicated by “hash” marks:


1
$N_{0,2}$
$N_{1,2}$
$N_{2,2}$
$N_{3,2}$
$N_{4,2}$
$N_{5,2}$
$N_{6,2}$
$N_{7,2}$
0
1
2
3
4
5

Using P2.1 we see that the basis functions are defined on the following knot spans:

$N_{0,2}$: {0, 0, 0, 1}

$N_{1,2}$: {0, 0, 1, 2}

$N_{2,2}$: {0, 1, 2, 3}

$N_{3,2}$: {1, 2, 3, 4}

$N_{4,2}$: {2, 3, 4, 4}

$N_{5,2}$: {3, 4, 4, 5}

$N_{6,2}$: {4, 4, 5, 5}

Thus, the word “multiplicity” can be understood in two ways:

- the multiplicity of a knot in the knot vector, and

- the multiplicity of a knot with respect to a specific basis function.

For example, $u = 0$ has multiplicity of 3 in the above knot vector U.

But with respect to $N_{0,2}$, $N_{1,2}$, $N_{2,2}$, and $N_{5,2}$, $u = 0$ has multiplicity 3, 2, 1, 0, respectively.

And, from P2.5, the continuity of these functions at $u = 0$ must be: $N_{0,2}$ discontinuous, $N_{1,2}$ C^0-continuous, $N_{2,2}$ C^1-continuous, and $N_{5,2}$ totally unaffected (i.e., $N_{5,2}$ and all its derivatives are zero at $u = 0$ from both sides).

Note that $N_{1,2}$ “sees” $u = 0$ as a double knot, and hence it is C^0-continuous. Similarly, $N_{2,2}$ “sees” all its knots with multiplicity 1; thus it is C^1-continuous everywhere.

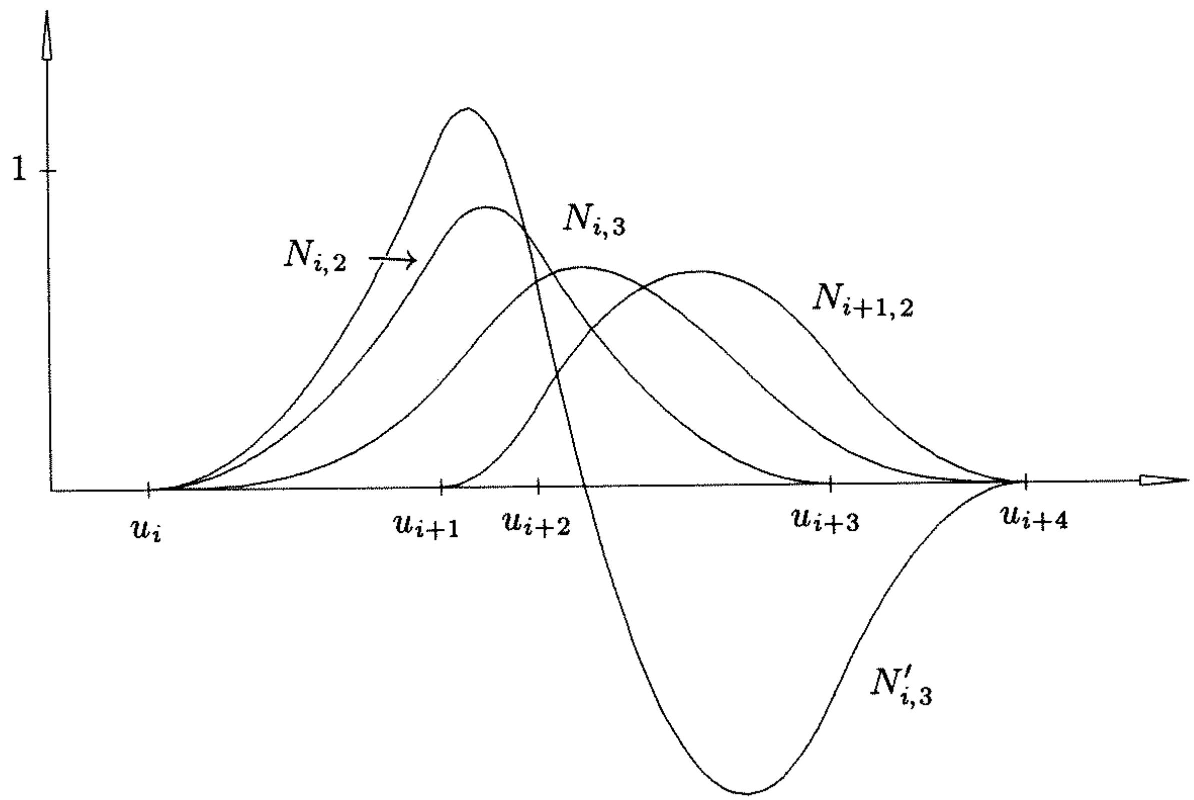
Another effect of multiple knots is to reduce the number of apparent intervals on which a function is non-zero. For example, $N_{6,2}$ is non-zero only on $u \in [4, 5)$, and it is only C^0-continuous at $u = 4$ and $u = 5$.

Derivatives of the B-spline Basis Functions

The derivative of a B-spline basis function is given by:

$$N'_{i,p}(u) = \frac{p}{u_{i+p} - u_i} N_{i,p-1}(u) - \frac{p}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)$$

(Proof by induction on *p*.) An example is illustrated below,



Repeated differentiation produces the general formula:

$$N_{i,p}^{(k)}(u) = p\left[\frac{N_{i,p-1}^{(k-1)}}{u_{i+p}-u_i} - \frac{N_{i+1,p-1}^{(k-1)}}{u_{i+p+1}-u_{i+1}}\right]$$

An alternative generalization computes the k-th derivative of $N_{i,p}(u)$ in terms of the functions $N_{i,p-k}$, ..., $N_{i+k,p-k}$: (assuming $k \leq p$, 0/0 = 0)

$$N_{i,p}^{(k)}(u) = \frac{p!}{(p-k)!} \sum_{j=0}^{k} a_{k,j} N_{i+j,p-k}$$

where

$$a_{0,0} = 1$$

$$a_{k,0} = \frac{a_{k-1,0}}{u_{i+p-k+1} - u_i}$$

$$a_{k,j} = \frac{a_{k-1,j} - a_{k-1,j-1}}{u_{i+p+j-k+1} - u_{i+j}}$$

$$where, \quad j = 1, \ldots, k-1$$

$$a_{k,k} = \frac{-a_{k-1,k-1}}{u_{i+p+1} - u_{i+k}}$$

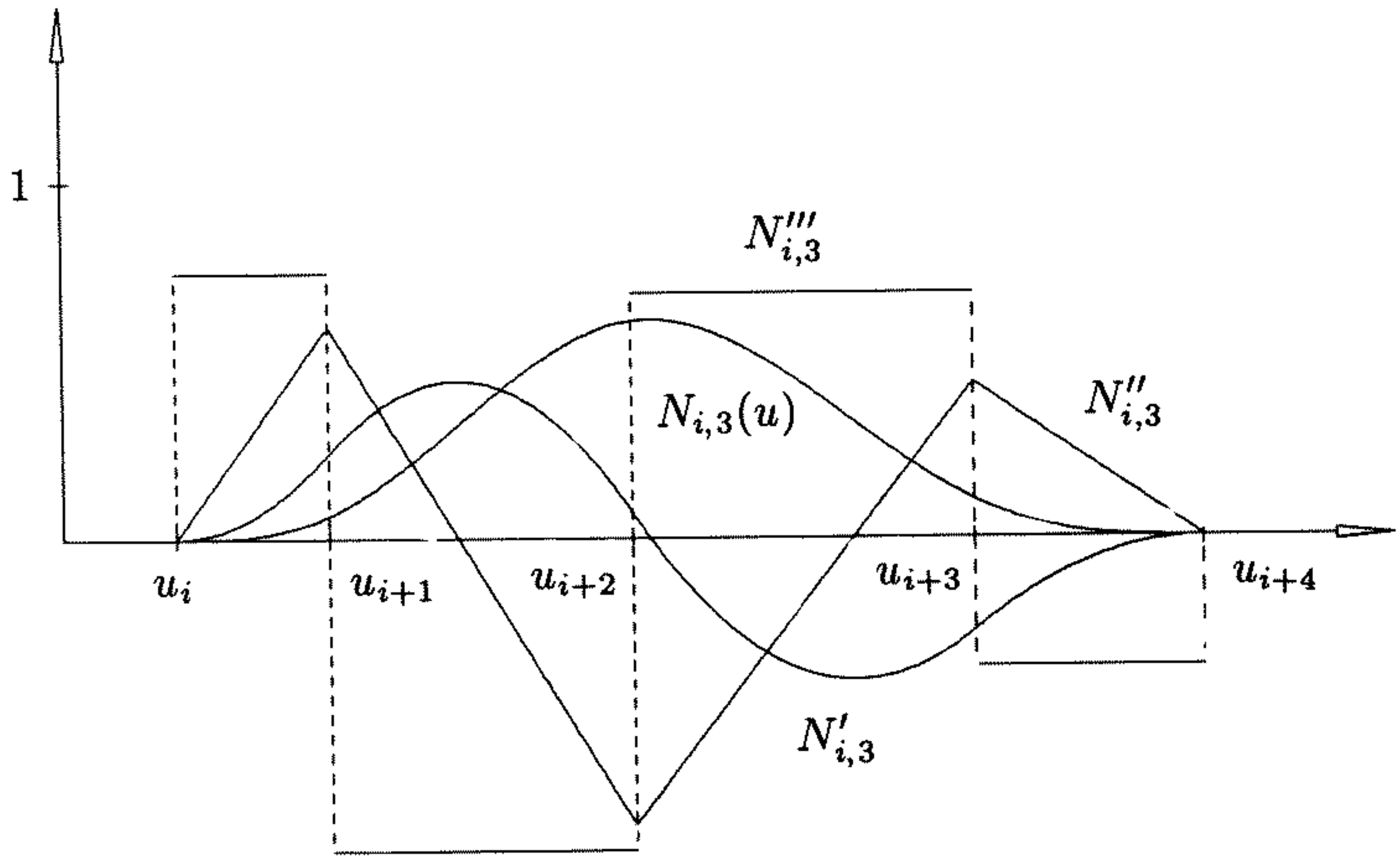
Finally, a third formulation:

$$N_{i,p}^{(k)} = \frac{p}{p-k}\left[\frac{u-u_i}{u_{i+p}-u_i}N_{i,p-1}^{(k)} + \frac{u_{i+p+1}-u}{u_{i+p+1}-u_{i+1}}N_{i+1,p-1}^{(k)}\right]$$

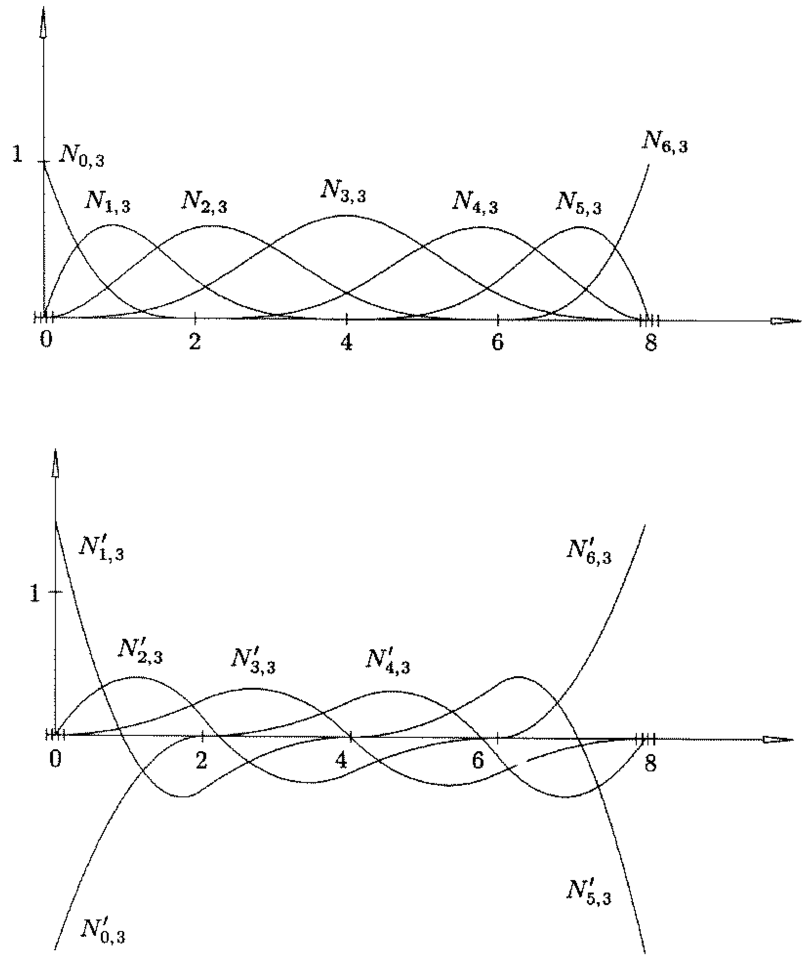
$$, k = 0, \ldots, p-1$$

gives the *k*-th derivative of $N_{i,p}(u)$ in terms of *k*-th derivative of $N_{i,p-1}$ and $N_{i+1,p-1}$

Example:

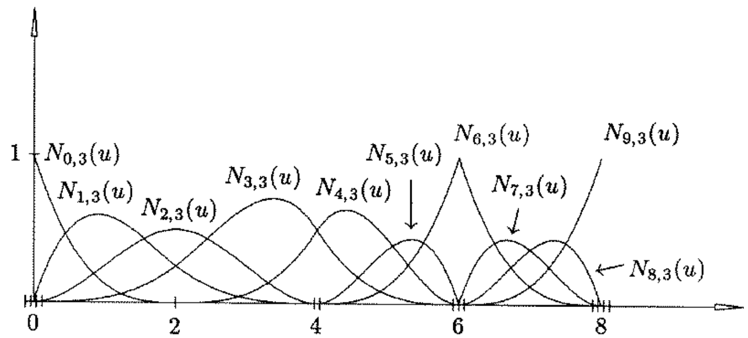


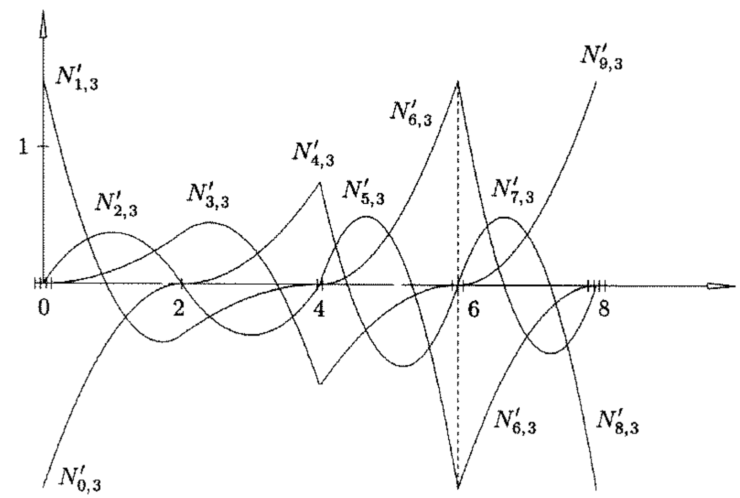
Example:



1
$N_{0,3}$
$N_{1,3}$
$N_{2,3}$
$N_{3,3}$
$N_{4,3}$
$N_{5,3}$
$N_{6,3}$
0
2
4
6
8
$N'_{1,3}$
$N'_{6,3}$
1
$N'_{2,3}$
$N'_{3,3}$
$N'_{4,3}$
0
2
4
6
8
$N'_{5,3}$
$N'_{0,3}$

Example:


1
$N_{0,3}(u)$
$N_{1,3}(u)$
$N_{2,3}(u)$
$N_{3,3}(u)$
$N_{4,3}(u)$
$N_{5,3}(u)$
$N_{6,3}(u)$
$N_{7,3}(u)$
$N_{8,3}(u)$
$N_{9,3}(u)$
0
2
4
6
8


1
$N'_{0,3}$
$N'_{1,3}$
$N'_{2,3}$
$N'_{3,3}$
$N'_{4,3}$
$N'_{5,3}$
$N'_{6,3}$
$N'_{7,3}$
$N'_{8,3}$
$N'_{9,3}$
0
2
4
6
8

Knot Vectors

Once the degree is fixed, the knot vector completely determines the functions $N_{i,p}$. There are several types of knot vectors, and unfortunately, terminology varies in the literature. We will consider only *non-periodic* knot vectors, which have the form:

$$U = \{ \underbrace{a, \ldots, a}_{p+1}, u_{p+1}, \ldots, u_{m-p-1}, \underbrace{b, \ldots, b}_{p+1} \}$$

For non-periodic knot vectors, the basis functions have two additional properties:

P2.7: A knot vector of the form,

$$U = \{ \underbrace{0, \ldots, 0}_{p+1}, \underbrace{1, \ldots, 1}_{p+1} \}$$

yields the Bernstein polynomials of degree p.

P2.8: Let $m + 1$ be the number of knots. Then there are $n + 1$ basis functions (and thus, $n + 1$ control points), where $n = m - p - 1$.

Alternatively if we want to use $n + 1$ control points, with a degree p B-spline curve (note that $n \geq p$) then the knot vector must have $m + 1 = n + p + 2$ entries.

Finally, given degree p and number of control points $n + 1$, the non-periodic B-spline will have $s = n - p + 1$ segments.

For the remainder of the course, all knot vectors are understood to be non-periodic.

Definition:

- A knot vector $U = \{ u_0, ..., u_m \}$ is said to be *uniform*, if all interior knots are equally spaced; i.e., if there exists a real number, d, such that, $d = u_{i+1} - u_i$ for all $p \leq i \leq m - p - 1$. Otherwise it is *non-uniform*.

Thus knot vectors with interior knot multiplicity greater than one are non-uniform.

Example: