Rational Bezier Surface

The perspective projection of a 4-dimensional polynomial Bezier surface,

$$\boldsymbol{S}^w(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,n}(u) B_{j,m}(v) \boldsymbol{P}^w{}_{ij}$$

and,

$$
\begin{aligned}
\boldsymbol{S}^w(u,v) &= H\{\boldsymbol{S}^w(u,v)\} \\
&= \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,n}(u)\,B_{j,m}(v)\,w_{ij}\boldsymbol{P}_{ij}}{\sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,n}(u)\,B_{j,m}(v)\,w_{ij}} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}(u,v)\,\boldsymbol{P}_{ij}
\end{aligned}
$$

where,

$$R_{i,j}(u,v) = \frac{B_{i,n}(u)\,B_{j,m}(v)\,w_{ij}}{\sum_{r=0}^{n}\sum_{s=0}^{m} B_{r,n}(u)\,B_{s,m}(v)\,w_{rs}}$$

Note that $R_{i,j}(u,v)$ are rational functions; but they are not products of other basis functions.

Hence, $\boldsymbol{S}(u,v)$ is not a tensor product surface (but $\boldsymbol{S}^{w}(u,v)$ is). We will generally work with $\boldsymbol{S}^{w}(u,v)$.

Exercise:

See if you can construct a rational Bezier surface patch representing a quarter-cylinder

B-spline Basis Functions

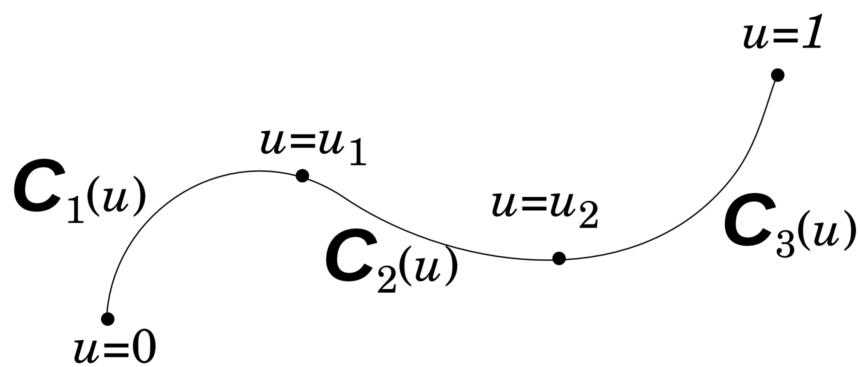
Curves consisting of a single polynomial or rational segment may have the following deficiencies:

- Degree must be high to satisfy a large number of constraints; e.g., need a degree (n-1) polynomial Bezier to pass through n data points ---> numerical instability.

- Degree must be high to fit complex shapes accurately
- Not suited to interactive shape design - shape control is not sufficiently localized.

One solution is to use curves (surfaces) which are *piecewise* polynomial or *piecewise* rational.

Consider the following curve



consisting of m (= 3) n-th degree polynomial segments. $\boldsymbol{C}(u)$ is defined on $u \in [0,1]$. The parameter values $u_0 = 0 < u_1 < u_2 < u_3 = 1$ are called breakpoints. Curve segments are denoted $\boldsymbol{C}_i(u)$, $i=0,...,m$.

The segments join with specified continuity (not necessarily the same at each breakpoint). Let $\boldsymbol{C}_i^{(j)}$ denote the j-th derivative of $\boldsymbol{C}_i$.

Definition:

Curve $\boldsymbol{C}(u)$ is said to be C^k-continuous at the breakpoint u_i, if

$$\boldsymbol{C}_i^{(j)}(u_i) = \boldsymbol{C}^{(j)}_{i+1}(u_i)$$
$$for\ \ all\ \ 0 \le j \le k$$

Note that any of the standard polynomial forms can be used to represent $\boldsymbol{C}_i(u)$.

Example:

Suppose the piecewise curve above is represented using cubic Bezier polynomials.

If the breakpoints are fixed, $U = \{u_0, u_1, u_2, u_3\}$, and the twelve control points $\boldsymbol{P}_i^j$ are allowed to vary arbitrarily, we obtain a vector space ϑ of all piecewise cubic polynomial curves on U.

Vector space ϑ has dimension 12, and a curve in ϑ can be discontinuous at u_1 or u_2.

Now suppose that C^0-continuity is specified, i.e., $\boldsymbol{P}_3^1 = \boldsymbol{P}_0^2$ and $\boldsymbol{P}_3^2 = \boldsymbol{P}_0^3$. Thus ϑ^0 is the vector space of all piecewise cubic polynomial curves on U which are at least C^0-continuous. Note that ϑ^0 has dimension 10, and $\vartheta^0 \subset \vartheta$.

Similarly, if C^1-continuity is specified, it can be shown that,

$$\boldsymbol{P}_3^1 = \frac{(u_2 - u_1)\boldsymbol{P}_2^1 + (u_1 - u_0)\boldsymbol{P}_1^2}{u_2 - u_0}$$

So ϑ^1 the vector space of all C^1-continuous piecewise polynomial curves on U has dimension 8, and $\vartheta^1 \subset \vartheta^0 \subset \vartheta$

Thus, storing and manipulating individual polynomial segments of a piecewise polynomial curve has the following deficiencies:

- redundant data must be stored - e.g., 12 coefficients, where only 8 are required for C^1-continuity, and 6 for C^2-continuity

- continuity of $\boldsymbol{C}(u)$ depends on the positions of the control points, e.g., in our previous example, suppose we like $\boldsymbol{C}_1(u)$ and $\boldsymbol{C}_3(u)$, but want to modify $\boldsymbol{C}_2(u)$ - if we want C^1-continuity throughout, we are stuck!

- Determining the continuity of a curve requires computation

We want a curve representation of the form,

$$\boldsymbol{C}(u) = \sum_{i=0}^{n} f_i(u)\,\boldsymbol{P}_i$$

where the $\boldsymbol{P}_i$ are control points and the $\{f_i(u),$ $i = 0, ..., n\}$ are piecewise polynomial functions forming a basis for the vector space of all piecewise polynomial functions of the desired degree and continuity...

i.e., continuity is determined by the basis functions, so control points can be modified *without* altering the curve's continuity.

Also, we seek basis functions $f_i(u)$ with *local support*, i.e., each $f_i(u)$ should be non-zero on only a limited number of subintervals (not the entire domain $[u_0, u_m]$).

B-spline Basis Functions

Definition (Cox-DeBoor):

Let $U = \{u_0, ..., u_m\}$ be a non-decreasing sequence of real numbers, i.e., $u_i \leq u_{i+1}$, $i = 0, ..., m - 1$. The u_i are called *knots*, and U is the *knot vector*. The i-th B-spline basis function of degree p (order $p + 1$), denoted as $N_{i,p}(u)$, is defined as follows:

$$N_{i,0}(u) = \begin{cases} 1 \quad if \quad (u_i \le u < u_{i+1}) \\ 0 \quad otherwise \end{cases}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)$$

Note the following:

- $N_{i,0}(u)$ is a step function, equal to zero everywhere except on the half-open interval $u \in [u_i, u_{i+1})$.
- For $p > 0$, $N_{i,p}(u)$ is a linear combination of two degree $p - 1$ basis functions.
- Computation of a set of basis functions requires specification of a knot vector U, and the degree, p.

- The formulation above can yield the quotient 0/0. We define this quotient to be zero.

- The half-open interval $[u_i, u_{i+1})$ is called the i-th *knot span*. It may have zero length since knots need not be distinct.

- Computation of the *p*-th degree functions generates a truncated triangular table,

$$
\begin{array}{cccc}
N_{0,0} & & & \\
 & N_{0,1} & & \\
N_{1,0} & & N_{0,2} & \\
 & N_{1,1} & & N_{0,3} \\
N_{2,0} & & N_{1,2} & \\
 & N_{2,1} & & N_{1,3} \\
N_{3,0} & & N_{2,2} & \vdots \\
 & N_{3,1} & \vdots & \\
N_{4,0} & \vdots & & \\
\vdots & & &
\end{array}
$$

Example:

Given U = { 0, 0, 0, 1, 1, 1 } compute the B-spline basis functions for degree p = 2

$$N_{0,0} = N_{1,0} = 0 \;\; for \;\; -\infty < u < \infty$$

$$N_{2,0} = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & otherwise \end{cases}$$

$$N_{3,0} = N_{4,0} = 0 \;\; for \;\; -\infty < u < \infty$$

$$N_{0,1} = \frac{u-0}{0-0}N_{0,0} + \frac{0-u}{0-0}N_{1,0}$$

$$= 0 \qquad -\infty < u < \infty$$

$$N_{1,1} = \frac{u-0}{0-0}N_{1,0} + \frac{1-u}{1-0}N_{2,0}$$

$$= \begin{cases} (1-u) & 0 \le u \le 1 \\ 0 & otherwise \end{cases}$$

$$N_{2,1} = \frac{u-0}{1-0}N_{2,0} + \frac{1-u}{1-1}N_{3,0}$$

$$= \begin{cases} u & 0 \le u \le 1 \\ 0 & otherwise \end{cases}$$

$$N_{3,1} = \frac{u-1}{1-1}N_{3,0} + \frac{1-u}{1-1}N_{4,0}$$

$$= 0 \qquad -\infty < u < \infty$$

$$N_{0,2} = \frac{u-0}{0-0}N_{0,1} + \frac{1-u}{1-0}N_{1,1}$$

$$= \begin{cases} (1-u)^2 & 0 \le u \le 1 \\ 0 & otherwise \end{cases}$$

$$N_{1,2} = \frac{u-0}{1-0}N_{1,1} + \frac{1-u}{1-0}N_{2,1}$$

$$= \begin{cases} 2(1-u) & 0 \le u \le 1 \\ 0 & otherwise \end{cases}$$

$$N_{2,2} = \frac{u-0}{1-0}N_{2,1} + \frac{1-u}{1-1}N_{3,1}$$

$$= \begin{cases} u^2 & 0 \le u \le 1 \\ 0 & otherwise \end{cases}$$

Note that the $N_{i,2}$ restricted to the interval $u \in [0,1]$, are the quadratic Bernstein polynomial.

Thus, the B-spline representation with knot vector of the form,

$$U = \{ \underbrace{0, \ldots, 0}_{p+1}, \underbrace{1, \ldots, 1}_{p+1} \}$$

is a generalization of the Bezier representation.