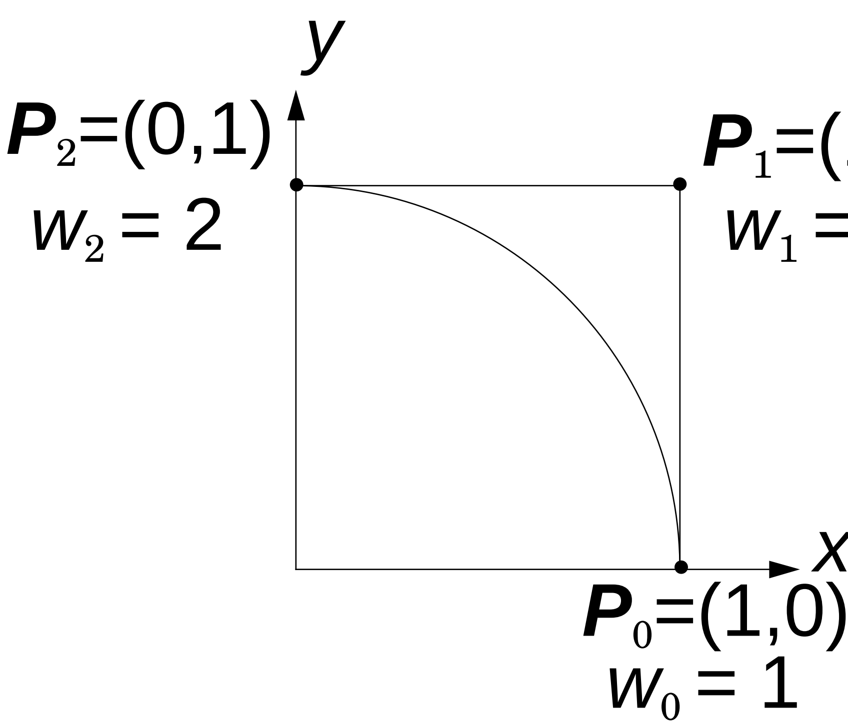
Conics and Circles

There are many important applications of conics and circles in design. Since the rational Bezier is a special case of a NURBS curve, quadric curves and surfaces can be represented exactly with the same representation used for general sculptured curves and surfaces. This is one of the fundamental advantages of the NURBS representation.

To motivate this discussion, we will consider the reparameterization of a circular arc, and then consider the construction of general conics in NURBS form.

Recall the rational Bezier of the form:

$\boldsymbol{P}^{w}_{i}$ -> (1, 0, 1), (1, 1, 1), (0, 2, 2)


y
$\mathbf{P}_2$=(0,1)
w_2 = 2
$\mathbf{P}_1$=(1,1)
w_1 = 1
x
$\mathbf{P}_0$=(1,0)
w_0 = 1

Evaluating the point at $u = 0.5$ yields (0.6, 0.8) which is obviously more than half the total arc length. This is to be expected from considering the derivatives (velocity vectors) at the endpoints, i.e., the curve is said to have non-uniform parameterization.

Suppose we want to reparameterize the curve in order to get a more uniform and symmetric parameterization.

A rational curve $\mathbf{C}(u)$ is reparameterized with a function of the form:

$$u = \frac{av + b}{cv + d}$$

This changes neither the shape nor the degree of the curve. Clearly, we may assume $d = 1$, and we want to satisfy the conditions: $u = 0$ at $v = 0$ and $u = 1$ at $v = 1$.

These conditions imply that: $b = 0$, and $c = a - 1$. Hence the reparameterization function we seek has the form:

$$u = \frac{av}{(a-1)v+1}$$

Now we must determine the coefficient a which yields a reparameterization that satisfies the following conditions:

$$\boldsymbol{C}\left(\frac{1}{2}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$and$$

$$\left|\dot{\boldsymbol{C}}(0)\right| = \left|\dot{\boldsymbol{C}}(1)\right|$$

Using the chain rule, $\dot{\boldsymbol{C}}(v) = \dot{\boldsymbol{C}}(u)\frac{du}{dv}$ and the fact that,

$$\dot{\boldsymbol{C}}(0) = 2\frac{w_1}{w_0}(\boldsymbol{P}_1 - \boldsymbol{P}_0)$$

$$\dot{\boldsymbol{C}}(1) = 2\frac{w_1}{w_2}(\boldsymbol{P}_2 - \boldsymbol{P}_1)$$

implies that,

$$2\left.\frac{du}{dv}\right|_{v\,=\,0} = \left.\frac{dv}{du}\right|_{v\,=\,1}$$

Differentiating the reparameterization relationship above yields,

$$\frac{du}{dv} = \frac{a}{[\,(a-1)\,v+1]^{2}}$$

Combining these and solving for *a* yields,

$$2a^2 - 1 = 0$$

or, $a = \pm\frac{\sqrt{2}}{2}$

Choosing $a = -\frac{\sqrt{2}}{2}$ would cause a zero in the denominator.

Hence, we choose $a = \frac{\sqrt{2}}{2}$, and obtain the reparameterization function,

$$u = \frac{\sqrt{2}v}{(\sqrt{2}-2)v+2}$$

Substituting this into the original curve definition $\boldsymbol{C}^w(u) = (1-u^2, 2u, 1+u^2)$ yields, for example,

$$x(v) = \frac{1 - \left[\dfrac{\sqrt{2}v}{(\sqrt{2}-2)\,v + 2}\right]^2}{1 + \left[\dfrac{\sqrt{2}v}{(\sqrt{2}-2)\,v + 2}\right]^2}$$

$$= \frac{(1-\sqrt{2})\,v^2 + (\sqrt{2}-2)\,v + 1}{(2-\sqrt{2})\,v^2 + (\sqrt{2}-2)\,v + 1}$$

Thus,

$$w(v) = (2-\sqrt{2})\,v^2 + (\sqrt{2}-2)\,v + 1$$

algebraic

$$= (1-v)^2 w_0 + 2v(1-v)\,w_1 + v^2 w_2$$

Bezier

It follows that:

$$w(0) = w_0 = 1$$

$$w(1) = w_2 = 1$$

and that at v = 1/2,

$$(2-\sqrt{2})\frac{1}{4} + (\sqrt{2}-2)\frac{1}{2} + 1 = \frac{1}{4} + \frac{1}{2}w_1 + \frac{1}{4}$$

and thus, $w_1 = \frac{\sqrt{2}}{2}$.

It is easy to verify that the Bezier representation given by

$\boldsymbol{P}^{w}{}_{i}$ -> (1, 0, 1), $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, (0, 1, 1) satisfies the desired parametric constraints, i.e.,

$$\boldsymbol{C}\left(\frac{1}{2}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$and$$

$$\left|\dot{\boldsymbol{C}}(0)\right| = \left|\dot{\boldsymbol{C}}(1)\right|$$

Some Important Things to Note

- Reparameterization of a rational curve with a function of the form: $u = \frac{av + b}{cv + d}$ changes neither the shape nor the degree.

- A reparameterization changes the weights. But arbitrarily changing the weights may change the shape of the curve. Hence there must exist a relationship among the weights, such that the shape is not changed as long as the relationship is not disrupted. This relationship depends solely on the degree of the curve. For quadratic Bezier curves, it is:

$$\frac{w_0 w_2}{w_1^2} = \text{constant}$$

This constant is called the conic shape factor. Changing the weights, while maintaining this equality, is equivalent to a reparameterization of the curve.

- A reparameterization (or equivalent change of weights) affects the *magnitude* (not direction) of the first derivative (velocity) vector

Conic Arcs

The quadratic rational Bezier curve,

$$\boldsymbol{C}(u) =$$

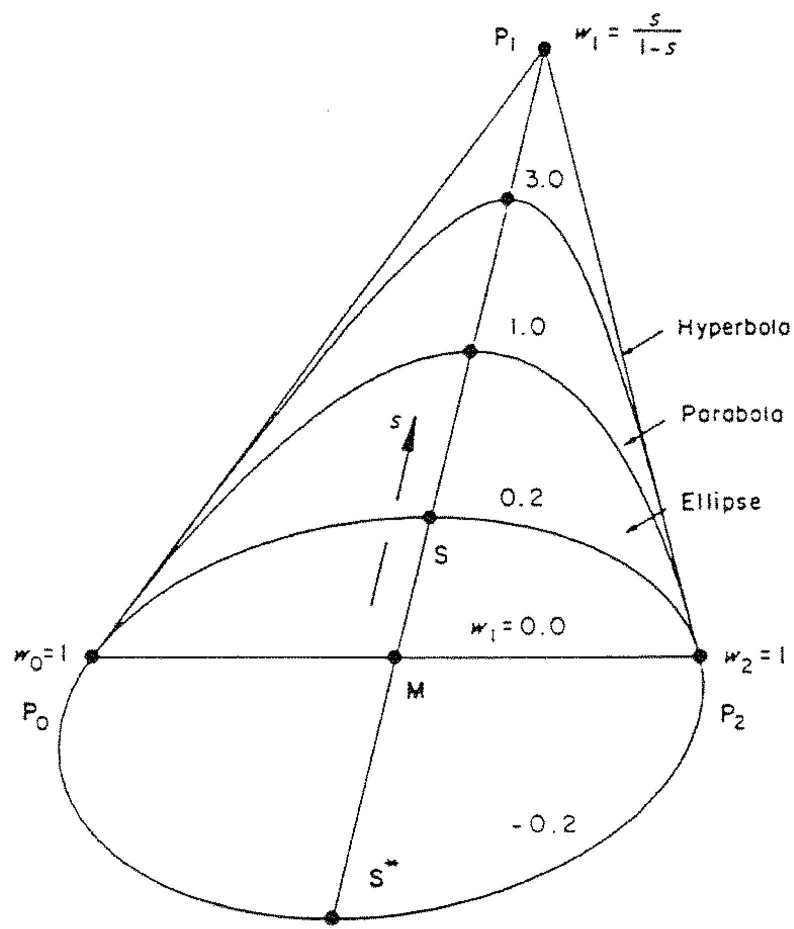
$$= \frac{(1-u)^2 w_0 \boldsymbol{P}_0 + 2u(1-u) w_1 \boldsymbol{P}_1 + u^2 w_2 \boldsymbol{P}_2}{(1-u)^2 w_0 + 2u(1-u) w_1 + u^2 w_2}$$

is a conic arc.

The denominator can be written,

$$w(u) = (1-u)^2 w_0 + 2u(1-u) w_1 + u^2 w_2$$

$$= ((w_0 - 2w_1 + w_2) u^2 + 2(w_1 - w_0) u + w_0)$$



P_1
$w_1 = \frac{s}{1-s}$
3.0
1.0
0.2
Hyperbola
Parabola
Ellipse
s
S
$w_1 = 0.0$
$w_0 = 1$
M
$w_2 = 1$
P_0
P_2
-0.2
S^*

The roots of this equation are:

$$u_{1,2} = \frac{w_0 - w_1 \pm \sqrt{1-k}}{w_0 - 2w_1 + w_2} \qquad \text{(A)}$$

where $k = \dfrac{w_0 w_2}{w_1^2}$ is the conic shape factor. If $w_i = 1$ for all i, then the rational Bezier is a parabola.

Let $w_0 = w_2 = 1$ (called the *normal parameterization*), assume $w_1 \neq 1$. Then the above equation implies:

- if $k > 1$, then equation (A) has no real solutions. There are no points at infinity on the curve; hence it is an ellipse.
- if $k = 1$ ($w_1 = -1$), equation (A) has one real solution; there is one point on the curve at infinity; thus the curve is a parabola.

- if $k < 1$, equation (A) has two roots, the curve has two points at infinity, it is a hyperbola.

Expressing the above conditions in terms of w_1, we have:

- $w_1^2 < 1$ ($-1 < w_1 < 1$) $\Rightarrow$ ellipse.
- $w_1^2 = 1$ ($w_1 = 1$ or -1) $\Rightarrow$ parabola.
- $w_1^2 > 1$ ($w_1 > 1$ or $w_1 < -1$) $\Rightarrow$ hyperbola.

Notice that w_1 can be 0 or negative. $w_1 = 0$ yields a straight line between $\boldsymbol{P}_0$ and $\boldsymbol{P}_2$, and $w_1 < 0$ yields the complementary arc (traversed in the reverse order). Notice also, that the convex hull property does not hold if $w_1 < 0$.

Varying w_1 yields a family of conic arcs having $\boldsymbol{P}_0$ and $\boldsymbol{P}_2$ as endpoints and end tangents parallel to $\boldsymbol{P}_0\boldsymbol{P}_1$ and $\boldsymbol{P}_1\boldsymbol{P}_2$.

A convenient way to select a conic from the family is top specify a third point on the conic, which is attained at some parameter value, say $u = 1/2$. This point is called the *shoulder point* of the conic: $\boldsymbol{S} = \boldsymbol{C}(1/2)$. Substitution of $u = 1/2$ into the equation of the rational quadratic Bezier yields:

$$\boldsymbol{S} = \frac{1}{1 + w_1}\boldsymbol{M} + \frac{w_1}{1 + w_1}\boldsymbol{P}_1$$

where $\boldsymbol{M}$ is the midpoint of the chord $\boldsymbol{P}_0\boldsymbol{P}_2$.

Due to our choice of $w_0 = w_2 = 1$, it follows that the tangent to the conic at S is parallel to $\boldsymbol{P}_0\boldsymbol{P}_2$, i.e., the conic attains its maximum distance from $\boldsymbol{P}_0\boldsymbol{P}_2$ at $\boldsymbol{S} = \boldsymbol{C}(1/2)$.

Let s be a new parameter that gives a linear interpolation between $\boldsymbol{M}$ and $\boldsymbol{P}_1$. Then for some value of s we have,

$$\boldsymbol{S} = (1-s)\boldsymbol{M} + s\boldsymbol{P}_1$$

Combining these two relationships yields,

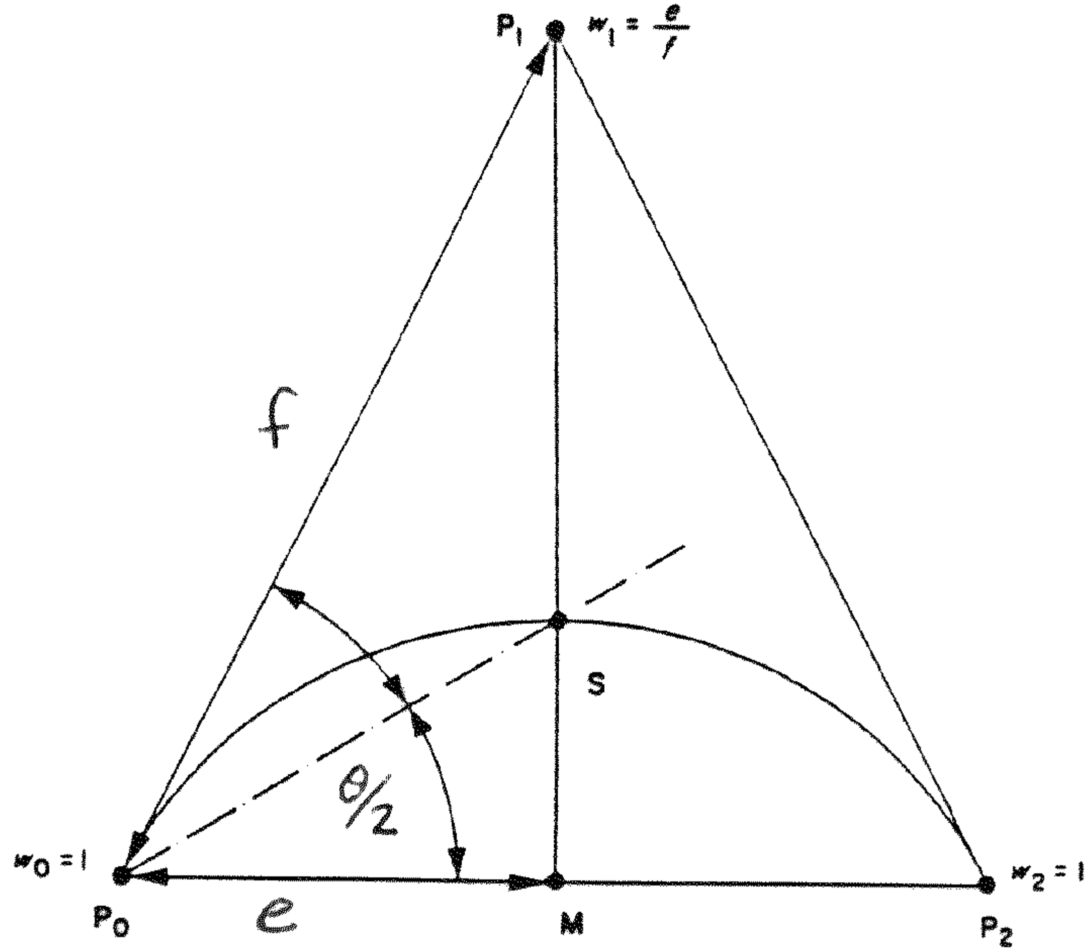
$$s = \frac{w_1}{1 + w_1} \qquad \text{and} \qquad w_1 = \frac{s}{1 - s}$$

Thus, the parameter *s* may be used as a design tool. The designer can move his shoulder point (which determines the “fullness” of the curve) linearly from $\boldsymbol{M}$ to $\boldsymbol{P}_1$ to yield:

- $s = 0$: a line segment
- $0 < s < 1/2$: an ellipse
- $s = 1/2$: a parabola
- $1/2 < s < 1$: a hyperbola

Circles

A circular arc whose sweep angle is less than 180° can also be represented by a rational quadratic Bezier curve.



P_1
$w_1 = \frac{e}{f}$
f
S
θ/2
$w_0 = 1$
P_0
e
M
P_2
$w_2 = 1$

For symmetry reasons, the triangle $\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2$ must be isosceles (with$|\boldsymbol{P}_0\boldsymbol{P}_1| = |\boldsymbol{P}_1\boldsymbol{P}_2|$).

Since a circle is a special case of an ellipse, we expect that $0 < w_1 < 1$. Using the linear relationship defined by s, we know that:

$$w_1 = \frac{s}{(1-s)} = \frac{|\boldsymbol{MS}|}{|\boldsymbol{SP}_1|}$$

Let $\theta = \angle \boldsymbol{P}_1\boldsymbol{P}_0\boldsymbol{M}$. It can be shown that the chord $\boldsymbol{P}_0\boldsymbol{S}$ bisects θ. Using the trig identity:

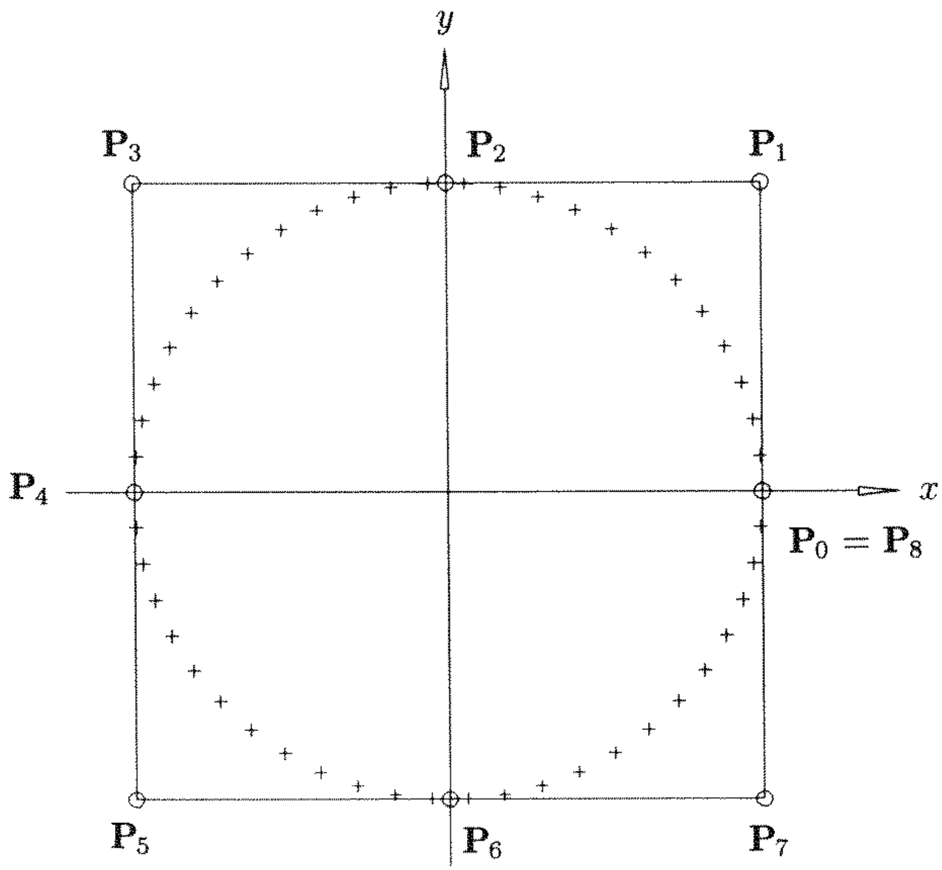
$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{(1 + \cos(\theta))}$, the above relationship, and the notation from the figure, we get,

$$w_1 = \frac{|\boldsymbol{MS}|}{|\boldsymbol{SP}_1|} = \frac{e \cdot \tan\left(\frac{\theta}{2}\right)}{f \cdot \sin(\theta) - \left(e \cdot \tan\left(\frac{\theta}{2}\right)\right)}$$

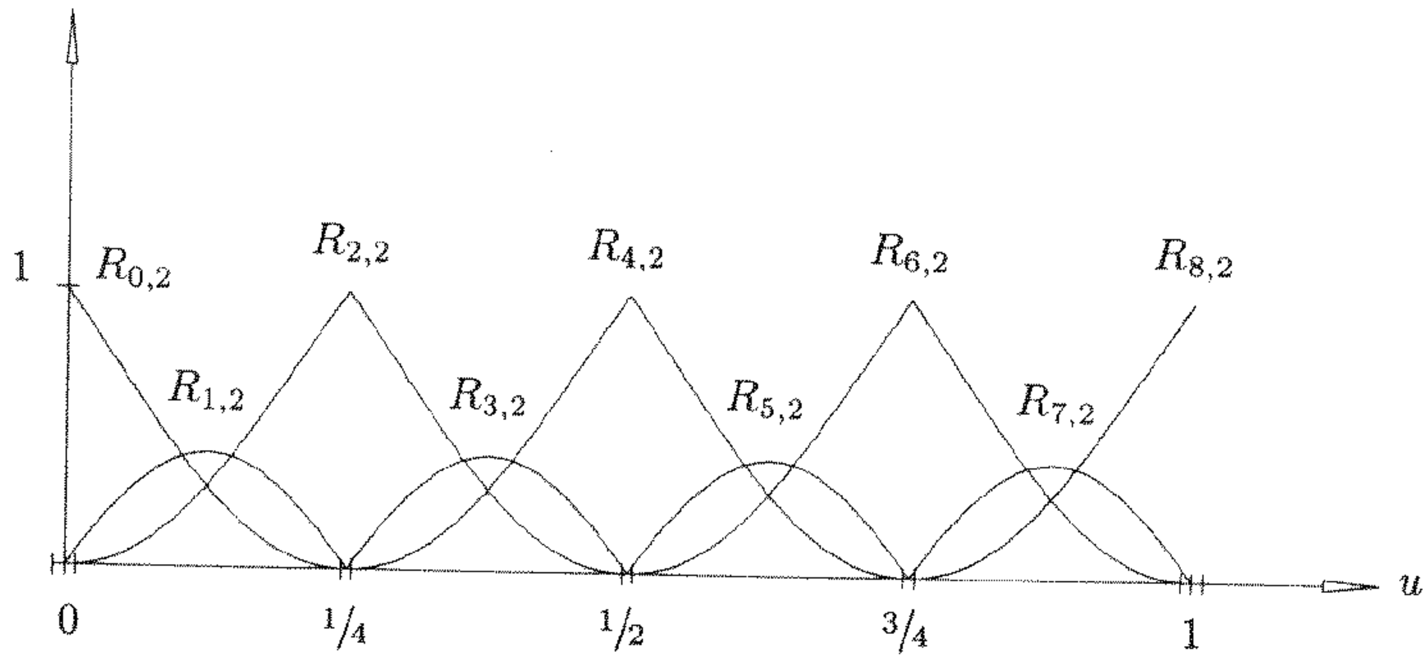
$$w_1 = \frac{e}{f(1 + \cos(\theta))}$$

$$= \frac{e}{f\left(1 - \frac{e}{f}\right) - e} = \frac{e}{f} = \cos(\theta)$$

The most convenient method for obtaining circular arcs equal to or greater than 180° is to piece together smaller arcs using multiple knots. Consider the following examples.

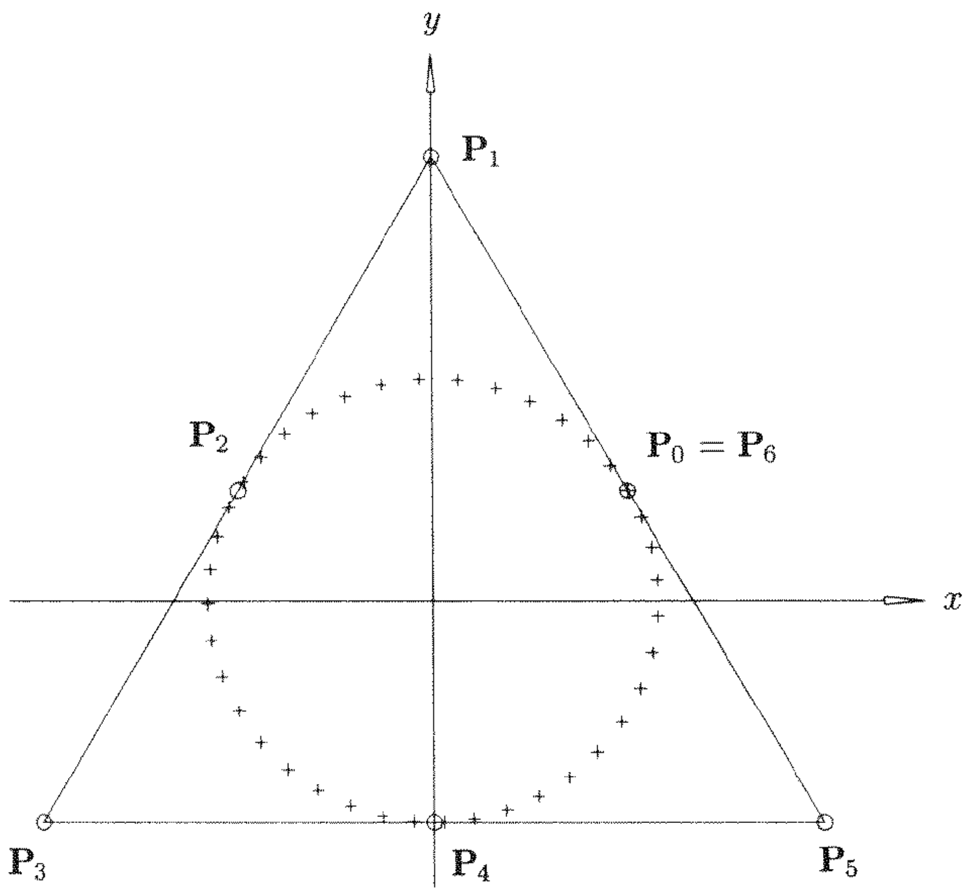


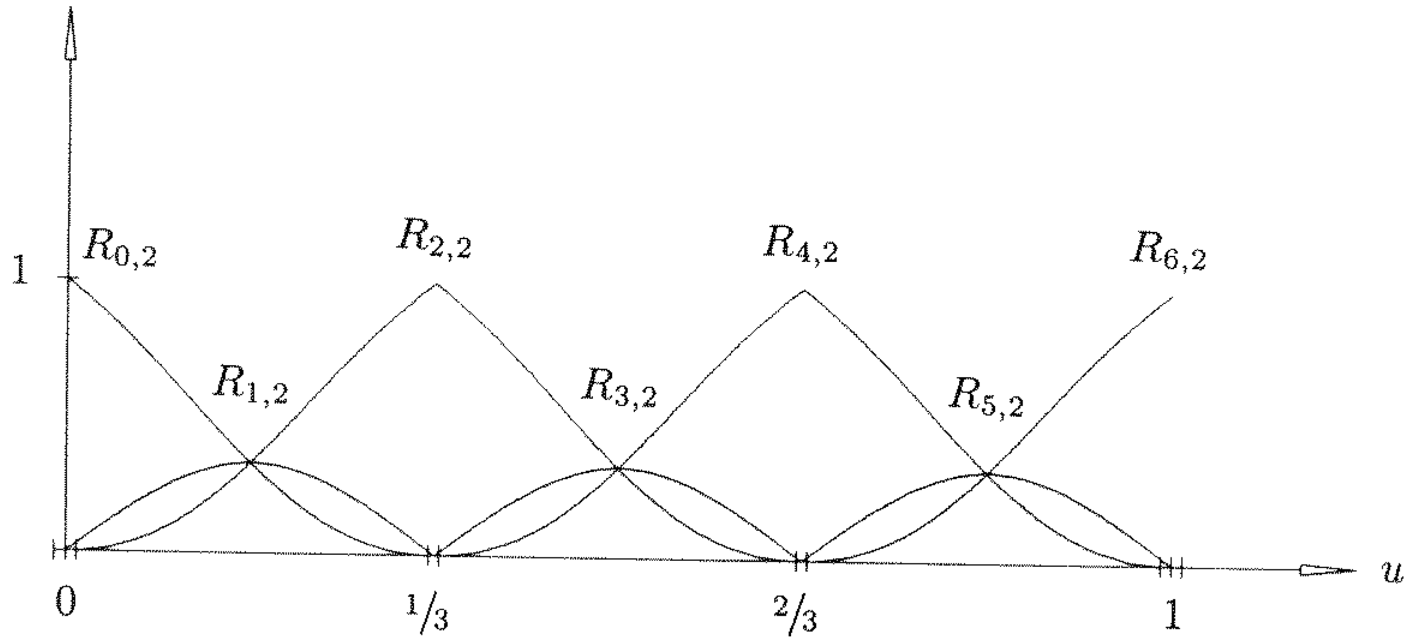
y
x
$\mathbf{P}_3$
$\mathbf{P}_2$
$\mathbf{P}_1$
$\mathbf{P}_4$
$\mathbf{P}_0 = \mathbf{P}_8$
$\mathbf{P}_5$
$\mathbf{P}_6$
$\mathbf{P}_7$



U={0 0 0 .25 .25 .5 .5 .75 .75 1 1 1}

W={1 $1/2^{.5}$ 1 $1/2^{.5}$ 1 $1/2^{.5}$ 1 $1/2^{.5}$ 1}


y
x
$\mathbf{P}_1$
$\mathbf{P}_2$
$\mathbf{P}_0 = \mathbf{P}_6$
$\mathbf{P}_3$
$\mathbf{P}_4$
$\mathbf{P}_5$



U={0 0 0 1/3 1/3 2/3 2/3 1 1 1}

W={1 .5 1 .5 1 .5 1}

There is a trade-off in deciding how many arcs to use for the full circle. The more arcs used, the better the parameterization and the tighter the convex hull.

Construction of Conics

This section will develop algorithms for construncting conics (using quadratic representations). Parabolic and hyberbolic arcs can always be represented with one rational Bezier curve and positive weights. However, as with circles, multi-segment curves may be required to obtain arbitrary elliptical arcs using positive weights.

There are many ways to specify a conic arc:

- Define parameters such as radii, axes and focal distance, as well as start and end points.
- Specify start and end points with tangent directions at those two points, plus one additional point on the arc.

If the conic arc is specify using data given in the form of item 1, then the start and end points ($\mathbf{P}_0$ & $\mathbf{P}_2$), the tangents at these points ($\mathbf{T}_0$ & $\mathbf{T}_2$) and a point on the conic arc (**P**) can be derived.

P1 can be obtained by intersecting lines [$\mathbf{P}_0$, $\mathbf{T}_0$] and [$\mathbf{P}_2$, $\mathbf{T}_2$]. Setting w_0=w_2=1, the only missing data is w_1.

The additional point, **P**, determines the conic and w_1. Substituting **P** = **C**(u) into:

$$\boldsymbol{C}(u) =$$

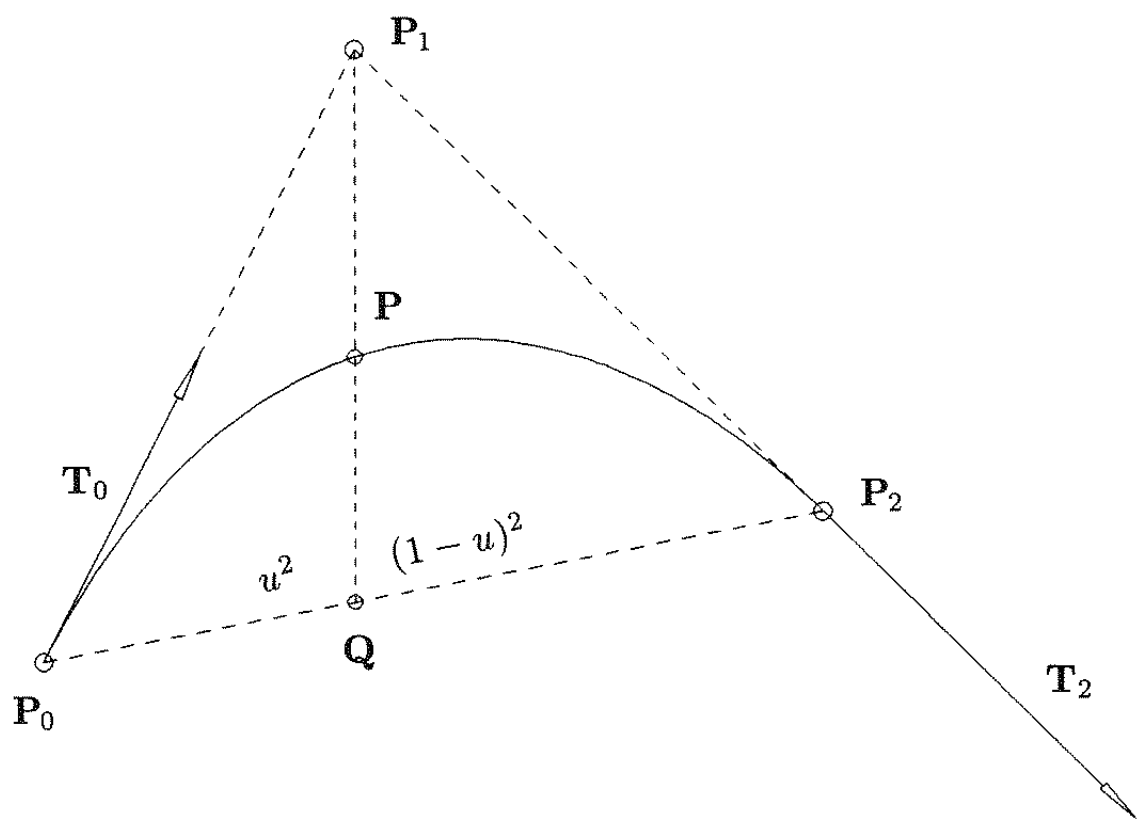
$$= \frac{(1-u)^2 w_0 \boldsymbol{P}_0 + 2u(1-u) w_1 \boldsymbol{P}_1 + u^2 w_2 \boldsymbol{P}_2}{(1-u)^2 w_0 + 2u(1-u) w_1 + u^2 w_2}$$

Yields three equations in the two knowns u and w_1.

These equations can be solved, but the geometric arguments yield a more efficient algorithm.

The desired conic can be considered as a perspective view of the parabola determine by $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$ with $\mathbf{P}_1$ being the center of the perspective. Any pair of conic segments lying in this triangle can be mapped onto one another.

This includes the line [$\mathbf{P}_0$, $\mathbf{P}_2$] onto the desired conic.



Hence the points **P** and **Q** are needed for this transformation. Now the line [$\mathbf{P}_0$, $\mathbf{P}_2$] is obtained by setting $w_1 = 0$.

$$\boldsymbol{L}(u) = \frac{(1-u)^2\boldsymbol{P}_0 + u^2\boldsymbol{P}_2}{(1-u)^2 + u^2}$$

L(*u*) is a convex conbination of $\mathbf{P}_0$ and $\mathbf{P}_2$, thus the ratio of distances |$\mathbf{P}_0$ **Q**| to |**Q** $\mathbf{P}_2$| is $u^2 : (1-u)^2$. This yields:

$$u = \frac{a}{1+a} \qquad a = \sqrt{\frac{|\boldsymbol{P}_0\boldsymbol{Q}|}{|\boldsymbol{Q}\boldsymbol{P}_2|}}$$

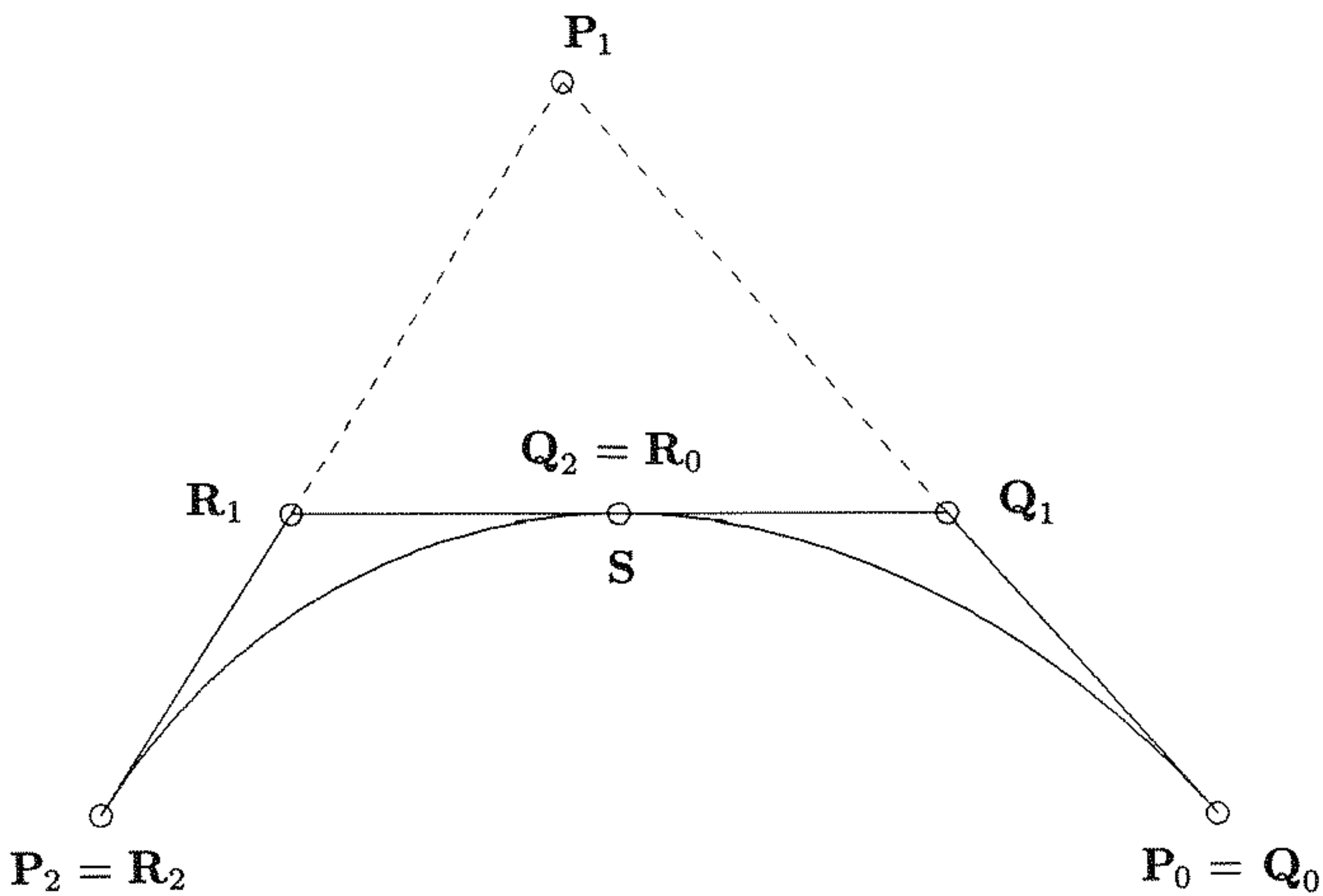
The weight, w_1, can now be found using u and **P**.

$$w_1 = \frac{(1-u)^2 (\boldsymbol{P}-\boldsymbol{P}_0) \cdot (\boldsymbol{P}_1-\boldsymbol{P}) + u^2 (\boldsymbol{P}-\boldsymbol{P}_2) \cdot (\boldsymbol{P}_1-\boldsymbol{P})}{2u(1-u)\,|\boldsymbol{P}_1-\boldsymbol{P}|^2}$$

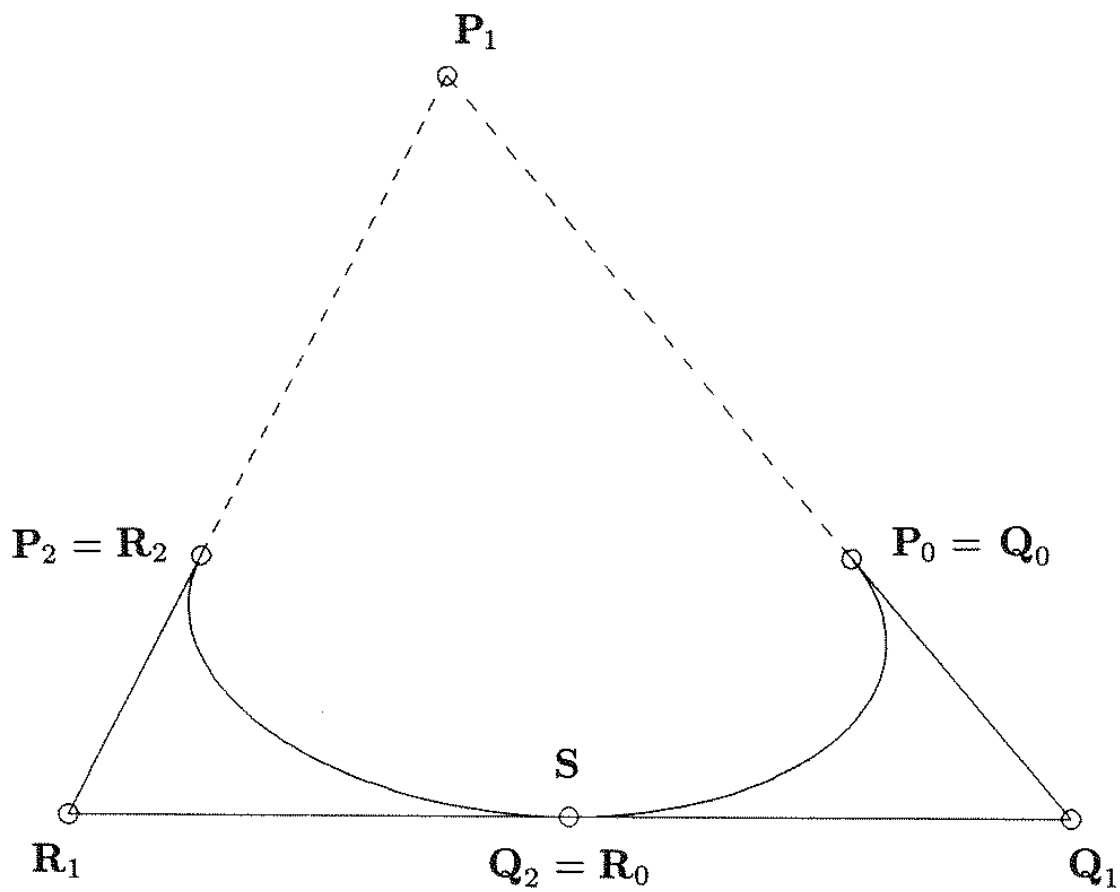
See algorithm A7.2.

Algorithm A7.2 will work for parabolic and hyperbolic arcs and for elliptical arcs for which $w_1 > 0$ and whose sweep angle is not too large. Splitting an ellipse into segments is not as easy as was the case for circles. The major and minor axes and radii are not available from the input data.

The shoulder point, **S**, is a convenient place to split the ellipitical arc.



$\mathbf{P}_1$
$\mathbf{Q}_2 = \mathbf{R}_0$
$\mathbf{R}_1$
$\mathbf{Q}_1$
$\mathbf{S}$
$\mathbf{P}_2 = \mathbf{R}_2$
$\mathbf{P}_0 = \mathbf{Q}_0$



$\mathbf{P}_1$
$\mathbf{P}_2 = \mathbf{R}_2$
$\mathbf{P}_0 = \mathbf{Q}_0$
$\mathbf{S}$
$\mathbf{R}_1$
$\mathbf{Q}_2 = \mathbf{R}_0$
$\mathbf{Q}_1$

The rational deCasteljau algorithm is used to split the arc $\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2$. There are two steps in the splitting process.

Step 1. Split at u = 0.5. Using the deCasteljau algorithm to obtain

$$\boldsymbol{Q}_1^{\,w} = \frac{1}{2}\boldsymbol{P}_0^{\,w} + \frac{1}{2}\boldsymbol{P}_1^{\,w}$$

$$\boldsymbol{R}_1^{\,w} = \frac{1}{2}\boldsymbol{P}_1^{\,w} + \frac{1}{2}\boldsymbol{P}_2^{\,w}$$

and recalling that $w_0 = w_2 = 1$, it follows that

$$\boldsymbol{Q}_1 = \frac{\boldsymbol{P}_0 + w_1 \boldsymbol{P}_1}{1 + w_1}$$

$$\boldsymbol{R}_1 = \frac{w_1 \boldsymbol{P}_1 + \boldsymbol{P}_2}{1 + w_1}$$

and

$$w_q = w_r = \frac{1}{2}(1 + w_1)$$

Where w_q and w_r are the weights at $\mathbf{Q}_1$ and $\mathbf{R}_1$. A second application of the deCasteljau algorithm yields.

$$\boldsymbol{R}_0 = \boldsymbol{Q}_2 = \boldsymbol{S} = \frac{1}{2}(\boldsymbol{Q}_1 + \boldsymbol{R}_1)$$

and

$$w_s = \frac{1}{2}(1 + w_1)$$

Step 2. Reparameterize so that the end weights are 1 for both of the two new segments. After splitting the weights for the first segment are:

$$w_0 = 1$$

$$w_q = \frac{1}{2}(1 + w_1)$$

$$w_s = \frac{1}{2}(1 + w_1)$$

The desired weights are:

$$w_0 = 1 \qquad w_{q1} \qquad w_{q2} = 1$$

Using the conic shape factor:

$$\frac{w_0 w_s}{w_q{}^2} = \frac{w_0 w_{q2}}{w_{q1}{}^2}$$

yields:

$$w_{q1} = \sqrt{\frac{1 + w_1}{2}}$$

Due to symmetry,

$$w_{r1} = \sqrt{\frac{1 + w_1}{2}}$$

See algorithm A7.3.