Important Properties of B-spline Basis Functions

P2.1 $N_{i,p}(u) = 0$ if $u$ is outside the interval $[u_i, u_{i+p+1})$ (local support property).

For example, note that $N_{1,3}$ is a combination of $N_{1,0}$, $N_{2,0}$, $N_{3,0}$, and $N_{4,0}$ Thus, $N_{1,3}$ is non-zero only on the interval $u \in [u_1, u_5]$. 
P2.2 In any given knot span, \([u_j, u_{j+1})\), at most 
\(p + 1\) of the \(N_{i,p}\) are non-zero, namely, the 
functions: \(N_{j-p,p}, \ldots, N_{j,p}\).
For example, on \([u_3, u_4]\) the only non-zero, 0-th degree function is \(N_{3,0}\). Hence the only cubic functions not zero on \([u_3, u_4]\) are \(N_{0,3}, \ldots, N_{3,3}\).
P2.3 \( N_{i,p}(u) \geq 0 \) for all \( i, p \), and \( u \) (Non-negativity). Can be proven by induction using P2.1.

P2.4 For arbitrary knot span, \([u_i, u_{i+1})\),

\[
\sum_{j = i - p}^{i} N_{j,p}(u) = 1 \quad \text{for all } u \in [u_i, u_{i+1})
\]

(Partition of unity)
P2.5 All derivatives of $N_{i,p}(u)$ exist in the interior of a knot span (where it is a polynomial). At a knot, $N_{i,p}(u)$ is $p - k$ times continuously differentiable, where $k$ is the multiplicity of the knot. Hence, increasing the degree increases continuity, and increasing knot multiplicity decreases continuity.

P 2.6 Except for the case of $p = 0$, $N_{i,p}(u)$ attains exactly one maximum value.
The concept of multiple knots is important.

Consider the example, $p = 2$, and $U = \{ 0, 0, 0, 1, 2, 3, 4, 4, 5, 5, 5 \}$. The basis function, $N_{7,2}$, are plotted below, with knot multiplicity indicated by “hash” marks:
Using P2.1 we see that the basis functions are defined on the following knot spans:

\[ N_{0,2} : \{0, 0, 0, 1\} \]
\[ N_{1,2} : \{0, 0, 1, 2\} \]
\[ N_{2,2} : \{0, 1, 2, 3\} \]
\[ N_{3,2} : \{1, 2, 3, 4\} \]
\[ N_{4,2} : \{2, 3, 4, 4\} \]
\[ N_{5,2} : \{3, 4, 4, 5\} \]
\[ N_{6,2} : \{4, 4, 5, 5\} \]
Thus, the word “multiplicity” can be understood in two ways:

- the multiplicity of a knot in the knot vector, and

- the multiplicity of a knot with respect to a specific basis function.

For example, $u = 0$ has multiplicity of 3 in the above knot vector $U$. 
But with respect to $N_{0,2}$, $N_{1,2}$, $N_{2,2}$, and $N_{5,2}$, $u = 0$ has multiplicity 3, 2, 1, 0, respectively.

And, from P2.5, the continuity of these functions at $u = 0$ must be: $N_{0,2}$ discontinuous, $N_{1,2}$ $C^0$-continuous, $N_{2,2}$ $C^1$-continuous, and $N_{5,2}$ totally unaffected (i.e., $N_{5,2}$ and all its derivatives are zero at $u = 0$ from both sides).
Note that $N_{1,2}$ “sees” $u = 0$ as a double knot, and hence it is $C^0$-continuous. Similarly, $N_{2,2}$ “sees” all its knots with multiplicity 1; thus it is $C^1$-continuous everywhere.

Another effect of multiple knots is to reduce the number of apparent intervals on which a function is non-zero. For example, $N_{6,2}$ is non-zero only on $u \in [4, 5)$, and it is only $C^0$-continuous at $u = 4$ and $u = 5$. 
Derivatives of the B-spline Basis Functions

The derivative of a B-spline basis function is given by:

\[ N'_{i,p}(u) = \frac{p}{u_{i+p} - u_i} N_{i,p-1}(u) \]

\[ -\frac{p}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \]
(Proof by induction on $\rho$.) An example is illustrated below,
Repeated differentiation produces the general formula:

\[ N_{i,p}^{(k)}(u) = p \left[ \frac{N_{i,p-1}^{(k-1)}}{u_{i+p} - u_i} - \frac{N_{i+1,p-1}^{(k-1)}}{u_{i+p+1} - u_{i+1}} \right] \]

An alternative generalization computes the \( k \)-th derivative of \( N_{i,p}(u) \) in terms of the functions \( N_{i,p-k}, \ldots, N_{i+k,p-k} \): (assuming \( k \leq p \), \( 0/0 = 0 \))
\[ N_{i,p}^{(k)}(u) = \frac{p!}{(p-k)!} \sum_{j=0}^{k} a_{k,j} N_{i+j,p-k} \]

where

\[ a_{0,0} = 1 \]

\[ a_{k,0} = \frac{a_{k-1,0}}{u_{i+p-k+1-u_i}} \]
\[ a_{k,j} = \frac{a_{k-1,j} - a_{k-1,j-1}}{u_{i+p+j-k+1} - u_{i+j}} \]

where, \( j = 1, \ldots, k-1 \)

\[ a_{k,k} = \frac{-a_{k-1,k-1}}{u_{i+p+1} - u_{i+k}} \]
Finally, a third formulation:

\[ N_{i,p}^{(k)} = \frac{p}{p-k} \left[ \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}^{(k)} + \frac{u_{i+p} - u}{u_{i+p} - u_{i+1}} N_{i+1,p-1}^{(k)} \right] \]

, \( k = 0, \ldots, p - 1 \)

Given the \( k \)-th derivative of \( N_{i,p}(u) \) in terms of \( k \)-th derivative of \( N_{i,p-1} \) and \( N_{i+1,p-1} \)
Example:
Example:
Example:
Knot Vectors

Once the degree is fixed, the knot vector completely determines the functions $N_{i,p}$. There are several types of knot vectors, and unfortunately, terminology varies in the literature. We will consider only non-periodic knot vectors, which have the form:

$$U = \{ a, \ldots, a, u_{p+1}, \ldots, u_{m-p-1}, b, \ldots, b \}$$

\[
\begin{array}{c}
p + 1 \\
\hline
\end{array}
\quad \begin{array}{c}
p + 1 \\
\hline
\end{array}
\]
For non-periodic knot vectors, the basis functions have two additional properties:

P2.7: A knot vector of the form,

\[ U = \{ \underbrace{0, \ldots, 0}_{p + 1}, 1, \ldots, 1 \} \]

yields the Bernstein polynomials of degree \( p \).
P2.8: Let $m + 1$ be the number of knots. Then there are $n + 1$ basis functions (and thus, $n + 1$ control points), where $n = m - p - 1$.

Alternatively if we want to use $n + 1$ control points, with a degree $p$ B-spline curve (note that $n \geq p$) then the knot vector must have $m + 1 = n + p + 2$ entries.

Finally, given degree $p$ and number of control points $n + 1$, the non-periodic B-spline will have $s = n - p + 1$ segments.
For the remainder of the course, all knot vectors are understood to be non-periodic.

Definition:

• A knot vector $U = \{ u_0, ..., u_m \}$ is said to be uniform, if all interior knots are equally spaced; i.e., if there exists a real number, $d$, such that, $d = u_{i+1} - u_i$ for all $p \leq i \leq m - p - 1$. Otherwise it is non-uniform.

Thus knot vectors with interior knot multiplicity greater than one are non-uniform.
Example: